INTERPOLATING BANACH-SAKS
AND DECOMPOSING OPERATORS

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Abstract With the same arguments of the work [H] this paper generalizes the main results of that: let I be one of the classes considered here, let $\bar{A}$, $\bar{B}$ be compatible pairs of Banach spaces, let $\bar{A}_{\theta,p}$, $\bar{B}_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ be the interpolation spaces obtained by the real method, $T : \bar{A} \to \bar{B}$ a bounded linear operator and T_{IS} the induced operator from the intersection $I(\bar{A})$ into the sum $S(\bar{B})$, then, the interpolated operator $T_{\theta,p}$ from $\bar{A}_{\theta,p}$ into $\bar{B}_{\theta,p}$ belongs to I if and only if $T_{IS} \in I$.

Introduction

Let $\bar{A}, \bar{B}$ be interpolation pairs and I be an operator ideal. One says that the ideal I possess the strong interpolation property for a method $\left(F\right)$ of interpolation (see [BL], chap. 2) if the interpolated operator $T_F : F\left(\bar{A}\right) \to F\left(\bar{B}\right)$ belongs to I when the induced $T_{IS} : I\left(\bar{A}\right) \to S\left(\bar{B}\right)$ is in I.

This paper generalizes a result of S. Heinrich [H] to prove that the classes of Banach-Saks and Decomposing operators possess the strong interpolation property with respect to the real method of interpolation depending on the parameters $0 < \theta < 1$ and $1 < p < \infty$.

1. Preliminaries For the concept of operator ideal and closed injective or surjective operator ideals see [H] or [P].

If $(X_m)_{m\in}$ is a sequence of Banach spaces denote by $(\sum_{m\in} \oplus X_m)_p$ with $1 \leq p < \infty$, the space of all sequences $(x_m)_{m\in}$ with $x_m \in X_m$ and

$$\|(x_m)_{m\in}\| = (\sum_{m\in} \|x_m\|^p_{X_m})^{1/p} < \infty.$$

Let J_i be the natural embedding of X_i into $(\sum_{m\in} \oplus X_m)_p$ and Q_j the projection of $(\sum_{m\in} \oplus X_m)_p$ onto X_j.The operator ideal I satisfies the $\sum_p$ condition if for any two sequences $(E_m)_{m\in}$ and $(F_m)_{m\in}$ of Banach spaces the following holds:

if $T \in L\left((\sum_{m\in} \oplus E_m)_p, (\sum_{m\in} \oplus F_m)_p\right)$ and $Q_j T J_i \in I(E_i, F_j)$ for every $i, , j \in$, then, $T \in I\left((\sum_{m\in} \oplus E_m)_p, (\sum_{m\in} \oplus F_m)_p\right)$.

Let $\bar{X} = (X_0, X_1)$ be an interpolation pair of Banach spaces, that is, X_0 and X_1 are Banach spaces imbedded into a Hausdorff topological vector space H. Denote by $I(\bar{X})$ the intersection $X_0 \cap X_1$ and by $S(\bar{X})$ the sum $X_0 + X_1$ with the usual norms.

Let $t > 0$; define for $x \in S(\bar{X})$ the K-functional

$$K(t, x) = \inf_{x = x_0 + x_1} (\|x_0\|_{X_0} + t\|x_1\|_{X_1})$$

and, for $x \in I(\bar{X})$, the J-functional

$$J(t, x) = \max(\|x\|_{X_0}, t\|x\|_{X_1}).$$

For $0 < \theta < 1$ and $1 \leq p < \infty$ the space $K_{\theta,p}(\overline{X}\ )$ is that of all $x \in S(\bar{X})$ for which

$$\left[\sum_{m\in}(2^{-\theta m}K(2^m, x))^p\right]^{\frac{1}{p}} < \infty.$$

The space $J_{\theta,p}(\overline{X}\ )$ consists of those $x \in S(\bar{X})$ for which there exists a sequence$(x_m)_{m\in}$ of $I(\bar{X})$ so that

$$x = \sum_{m\in} x_m \text{ (convergence in } S(\bar{X}))$$

with

$$\left[\sum_{m\in}(2^{-\theta m}J(2^m, x_m))^p\right]^{\frac{1}{p}} < \infty.$$

In each case the norm is

$$\|x\|_{\theta,p;K} = \left[\sum_{m\in}(2^{-\theta m}K(2^m, x))^p\right]^{\frac{1}{p}}$$

and

$$\|x\|_{\theta,p;J} = \inf_{\sum x_m = x}\left[\sum_{m\in}(2^{-\theta m}J(2^m, x_m))^p\right]^{\frac{1}{p}}.$$

The spaces $K_{\theta,p}(\overline{X}\ )$ and $J_{\theta,p}(\overline{X}\ )$ are interpolation spaces with respect to $\overline{X}$; they are in fact equal and the norms equivalent ([BL]). Accordingly, anyone of them will be the space $\overline{X}_{\theta,p}$ of the real method of interpolation.

Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two interpolation pairs. If $T : S(\bar{A}) \to S(\bar{B})$ is a bounded linear operator whose restriction to A_i is bounded from A_i into B_i $(i = 0, 1)$ one says that T is a bounded operator from $\bar{A}$ into $\bar{B}$ and write $T : \bar{A} \to \bar{B}$. In this case the interpolated operator from $\bar{A}_{\theta,p}$ into $\bar{B}_{\theta,p}$ is bounded; it will be denoted by $T_{\theta,p}$.

An operator $T \in L(E, F)$ is a Banach-Saks operator if any bounded sequence (x_n) of E has a subsequence (x_{n_k}) such that (Tx_{n_k}) is Césaro convergent. $T \in L(E, F)$ is a decomposing operator if T^* is Radon-Nikodym; see [P], chap. 24.

The classes of weakly compact, separable, Rosenthal and unconditionally summing operators are well known.

The ideal of weakly compact operators will be denoted by W, by X that of separable, by R Rosenthal, by U unconditionally summing, by S Banach-Saks and by Q the ideal of decomposing operators.

In previous works, see [MQ] and [R] Thm. 10.10, it has been proved the following theorem (in [R] for a more general method of interpolation)

Theorem 1.1 The ideals W, R and X possess the strong interpolation property for the real method with $0 < \theta < 1$ and $1 < p < \infty$.

2. Interpolation of Banach-Saks and Decomposing Operators Let $\bar{A}, \bar{B}$ be interpolation pairs. In order to avoid a complicated notation write A for the intersection $I(\bar{A})$ and B for the sum $S(\bar{B})$. Define on A and B the following equivalent norms

$$\begin{aligned} \|x\|_m &= 2^{-\theta m} J\left(2^m, x\right) \qquad \text{for } x \in A \text{ and } m \in \\ \|y\|_m &= 2^{-\theta m} K\left(2^m, x\right) \qquad \text{for } y \in B \text{ and } m \in . \end{aligned}$$

Denote by A_m the space $(A, \| \quad \|_m)$ and by B_m the space $(B, \| \quad \|_m)$. For each $(x_m)_{m\in} \in \left(\sum_{m\in} \oplus A_m\right)_p$ the sum $\sum_{m\in} x_m$ converges in $S(\bar{A})$. Then, there is a surjection Q from $\left(\sum_{m\in} \oplus A_m\right)_p$ onto $\bar{A}_{\theta,p} = J_{\theta,p}\left(\bar{A}\right)$ with $0 < \theta < 1$ and $1 < p < \infty$

$$Q\left((x_m)_{m\in}\right) = \sum_{m\in} x_m \qquad (\text{convergence in } S\left(\bar{A}\right))$$

and an isomorphic embedding J from $\bar{B}_{\theta,p} = K_{\theta,p}\left(\bar{B}\right)$ into $\left(\sum_{m\in} \oplus B_m\right)_p$ defined by $J(y) = (\ldots, y, y, y, \ldots)$.

Proposition 2.1 If I is a (closed) injective and surjective operator ideal which satisfies the $\sum_p$ condition it possess the strong property of interpolation.

Proof Let J_i be the embedding of A_i into $\left(\sum_{m\in} \oplus A_m\right)_p$ and Q_j the projection of $\left(\sum_{m\in} \oplus B_m\right)_p$ onto B_j. Obviously the operator $Q_j JTQJ_i$ is the operator T_{IS} from $A_i = I\left(\bar{A}\right)$ into $B_j = S\left(\bar{B}\right)$. It is, then, an operator of the class I. Since I satisfies the $\sum_p$ condition the operator JTQ belongs to $I\left(\left(\sum_{m\in}\oplus A_m\right)_p, \left(\sum_{m\in}\oplus B_m\right)_p\right)$. Now, the injectivity and surjectivity of I implies that $T : \bar{A}_{\theta,p} \to \bar{B}_{\theta,p}$ is in I, that is, $T_{\theta,p} \in I\left(\bar{A}_{\theta,p}, \bar{B}_{\theta,p}\right)$ End Proof

Theorem 2.2 The ideals S and Q possess the strong interpolation property for the real method with $0 < \theta < 1$ and $1 < p < \infty$.

Proof The ideals S and Q satisfy the $\sum_p$ condition, see [H], pages 407 to 409; since they are injective and surjective, Proposition 2.1 apliesEnd Proof

Remark 2.4 Theorem 2.2 is not true neither for $p = 1$ nor for $p = \infty$. Indeed, according to M. Levy (see [L]), the interpolation spaces $\bar{X}_{\theta,1}$ and $\bar{X}_{\theta,\infty}$ contain, in the non trivial case, (complemented) isomorphic copies of l_1 and l_∞ respectively.

3. Interpolation of Dual Ideals Some order relations, not stated in [P], are collected in the next proposition; they are easy extensions of well known facts in Banach space theory

Proposition 3.1 (i)$R \subset U^{dual}$, (ii) $X^{dual} \subset U^{dual}$ and (iii) $R^{dual} \subset U$. All inclusions are strict.

Lemma 3.2 Let I be a surjective operator ideal with the strong interpolation property for the real method (depending on $0 < \theta < 1$ and $1 < p < \infty$). Then, I^{dual} also possesses the strong interpolation property.

Proof One can suppose that $I\left(\bar{A}\right)$ is dense in A_i and that $I\left(\bar{B}\right)$ is dense in B_i for $i = 0, 1$ because $\bar{B}_0 + \bar{B}_1$ is a closed subspace of $S\left(\overline{B}\right)$ and, by the injectivity, T_{IS} is in I^{dual} if and only if $T : I\left(\bar{A}\right) \to \bar{B}_0 + \bar{B}_1$ is in I^{dual}. If $T^* : I\left(\bar{B}^*\right) \to S\left(\overline{A}^*\right)$ is in I then, by the duality theorem $T^*_{\theta,p} : \bar{B}^*_{\theta,p} \to \bar{A}^*_{\theta,p}$ is in I for $1 < p < \infty$ and thus $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ is in I^{dual}End Proof

It follows at once,

Theorem 3.3 The ideals $X^{dual}, R^{dual}, S^{dual}$ and Q^{dual} possess the strong interpolation property (real method with $0 < \theta < 1$ and $1 < p < \infty$).

Corollary 3.4 Let $\bar{X}$ be an interpolation pair. Then, the space $\bar{X}^*_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ is separable, has no isomorphic copy of l_1, possesses the Banach-Saks property or its dual has the Radon-Nikodym property if and only if the imbedding i from $I(\bar{X})$ into $S\left(\bar{X}\right)$ belongs to $X^{dual}, R^{dual}, S^{dual}$ or Q^{dual} respectively.

4. Final Notes It is well known that every surjective operator ideal with the strong interpolation property for the real method (depending on $0 < \theta < 1$ and $1 < p < \infty$) has the factorization property, see [H]. Then,

Theorem 4.2 The operator ideals $X^{dual}, R^{dual}, S^{dual}$ and Q^{dual} possess the factorization property.

Suppose that A and B are closed, injective and surjective operator ideals with the strong interpolation property for the real method ($0 < \theta < 1$ and $1 < p < \infty$). Since $B \circ A$ is the intersection $B \cap A$ (see [H]), this ideal also possess the strong interpolation property. On the other hand the strong interpolation property is preserved by dualizing (Lemma 3.2).

The classes X,R,W,S and Q are closed, injective, surjective and possess the strong interpolation property. Therefore, taking products and dualizing one obtain many new classes of injective, surjective and closed operator ideals with the strong interpolation property for the real method ($0 < \theta < 1$ and $1 < p < \infty$) and, by that, with the factorization property.

Take non trivial chains, as for example $W \circ X$ or $Q \circ (R \circ X)^{dual}$; they possess the strong interpolation property and the factorization property.

Perhaps is interesting the following proof of the fact that $W \circ X$ is completely symmetric

Proposition 4.3 $W \circ X = (W \circ X)^{dual}$.

Proof. $W \circ X$ and $(W \circ X)^{dual}$, both, posses the factorization property. Since a Banach space is reflexive and separable if and only if its dual is reflexive and separable one concludesEnd Proof

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