FACTORIZATION OF MIXED OPERATORS

Alexi Quevedo
UCV, Caracas

In memory of my beloved Father, Bernardo

Abstract

Let T be an operator between Banach spaces that is, for example, separable, Rosenthal, and decomposing. The real method of interpolation of Lions-Peetre, for pairs, is used to prove that T factors through a separable Banach space S that has no subspace isomorphic to ℓ_1 and whose dual S^* has the Radon-Nikodým property. A technique to produce such factorization spaces for *mixed operators* is introduced. For this, it is necessary first to prove that many *mixed operator ideals* possess the *strong property of interpolation* for the real method of Lions-Peetre.

Key words and phrases: Banach spaces, factorization of operators, real method of interpolation.

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§1. Introduction.

After 1974, when W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński announced their result on factorization of weakly compact operators, see [4], B. Beauzamy and S. Heinrich, using the real method of interpolation of J. L. Lions and J. Peetre, produced more results of this type. They proved that the operator ideals of Rosenthal, Banach-Saks, alternate-sign Banach-Saks and decomposing operators have the *factorization property*. In this paper, following a technique developed by Heinrich in [7], first, it is proved that the referred operator ideals, their dual ideals and many other operator ideals (*mixed operator ideals*), satisfy the (so called, for lack of a better name), *strong property of interpolation*, with regard to the Lions-Peetre method of interpolation, and then, following a technique proposed by Beauzamy in [1, page 37] and by Heinrich in [7, page 406], a Theorem of factorization for *mixed operators* is obtained.

A suitable example to understand the factorization of a mixed operator, is that of an operator between Banach spaces, which is, at the same time, weakly compact and separable: this operator factorizes through a reflexive and separable Banach space.

The main result of the paper is Corollary 4.2 of **§4**, which is obtained after a much needed introduction. In Theorem 3.4 of **§3**, a necessary result on interpolation is proved.

We employ the standard notation, see [3], [7], or [9]. Throughout the paper E and F will denote Banach spaces.

In order to obtain a short paper, it has been supposed that such concepts as operator ideals, injective, surjective and closed operator ideals, are known. For the notion of the $\sum_p$-condition, $1 \leq p < \infty$, see [7, Section II]. Also, a detailed explanation of the Lions-Peetre real method of interpolation has been avoided; in the book of J. Bergh and J. Löfström, [3, Chapter 3], an excellent and complete description of that method is given.

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§2. Mixed Operators.

2.1. Some operator ideals. The classes of separable, weakly compact, Rosenthal, Banach-Saks, alternate-signs Banach-Saks, Banach-Saks-Rosenthal (or weak Banach-Saks), Radon-Nikodým and decomposing operators are well known, see [7, page 406], [2] and [8]. All of them are operator ideals according to A. Pietsch and, like in his book, [9], Gothic capital letters will represent each one of them. Thus, $\mathfrak{X}$ is the ideal of separable operators; $\mathfrak{W}$, weakly compact operators; $\mathfrak{R}$, Rosenthal operators; $\mathfrak{Y}$, Radon-Nikodým operators and $\mathfrak{Q}$, decomposing operators.

The operator ideals of Banach-Saks, alternate-signs Banach-Saks and that of Banach-Saks-Rosenthal, are not treated by Pietsch in his book. They will be represented by $\mathcal{BS}$, $\mathcal{ABS}$ and $\mathcal{BSR}$, respectively (the notation is of Beauzamy); see [2] and [8] for their study.

All of these operator ideals are closed and injective; $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}$ and $\mathfrak{Q}$ are also surjective. Neither $\mathfrak{Y}$ nor $\mathcal{BSR}$ is surjective.

2.2. Mixed operator ideals. If $\mathfrak{C}$ and $\mathfrak{D}$ are two operator ideals, the intersection $\mathfrak{C} \cap \mathfrak{D}$ is a new operator ideal that inherits many good properties of $\mathfrak{C}$ and $\mathfrak{D}$. For example, if they are closed, so is the intersection; if they satisfy the $\sum_p$-condition for some $p \in [1, \infty)$, the intersection also satisfies it. Thus some good properties of an operator can be established once it is known that it is a mixed operator, i.e., it belongs to the intersection of some operator ideals with good properties.

Let $\mathfrak{D} \circ \mathfrak{C}$ be the product of $\mathfrak{C}$ and $\mathfrak{D}$, in the sense given to it by Pietsch in [9, 3.1]. It is always true that $\mathfrak{D} \circ \mathfrak{C} \subset \mathfrak{D} \cap \mathfrak{C}$ but the converse inclusion is not true in general. Heinrich, by using interpolation theorems also due to him, see [7, Theorem 1.3], found that if $\mathfrak{C}$ is injective, $\mathfrak{D}$ is surjective and both are closed, then $\mathfrak{D} \circ \mathfrak{C} = \mathfrak{D} \cap \mathfrak{C}$; see also [9, Epilogue, page 407]. It should be remarked that in most discussed cases of this paper intersections of operator ideals can be seen as products.

Let $\mathfrak{I}$ be an operator ideal. The operator $T \in \mathfrak{L}(E,F)$ belongs to the *dual ideal* $\mathfrak{I}^{dual}(E,F)$ if the adjoint operator, T^*, belongs to $\mathfrak{I}(F^*,E^*)$, see [9, 4.4]. For example, for $\mathfrak{Q}$ one has that $\mathfrak{Q} = \mathfrak{Y}^{dual}$, see [9, Theorem 24.4.3].

If $\mathfrak{I}$ is injective, $\mathfrak{I}^{dual}$ is surjective and if $\mathfrak{I}$ is surjective, $\mathfrak{I}^{dual}$ is injective, see [9, 4.7.18]. If $\mathfrak{I}$ is closed, $\mathfrak{I}^{dual}$ is closed. If $\mathfrak{I}$ satisfies the $\sum_p$-condition for all $p \in (1,\infty)$ then $\mathfrak{I}^{dual}$ satisfies the $\sum_p$-condition for all $p \in (1,\infty)$.

Operator ideals as $\mathfrak{M} = \mathfrak{X} \cap \mathfrak{W}$, $\mathfrak{M} = \mathfrak{R} \cap \mathfrak{R}^{dual}$, $\mathfrak{M} = (\mathfrak{Q} \cap \mathcal{ABS})^{dual}$ or $\mathfrak{M} = \mathfrak{X} \cap \mathfrak{R}^{dual} \cap \mathfrak{Q}$, being, all of them injective, surjective and closed, are examples of mixed operator ideals.

2.3. Relations between the Banach-Saks operator ideals. The relations $\mathcal{BS} \subset \mathcal{ABS} \subset \mathcal{BSR}$, with strict inclusions, are well known, see e.g., [2, page 373] or [8, §1].

The following equality simply extends the respective theorem of Beauzamy on Banach spaces with the alternate-signs Banach-Saks property (see [2]) to the operator ideal $\mathcal{ABS}$.

Theorem 2.3. It holds that $\mathcal{ABS} = \mathfrak{R} \cap \mathcal{BSR} = \mathfrak{R} \circ \mathcal{BSR}$ (product in this order since $\mathcal{BSR}$ is not surjective).

Proof. Proceed as Beauzamy does in [2, Section II, page 362] and define the property $(\mathcal{P}_1)$ for operators: an operator $T : E \to F$ has the property $(\mathcal{P}_1)$ if there exists $\delta > 0$ and a bounded sequence (x_n) in E such that for all $k \in \mathbb{N}$, any choice of signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k = \pm 1$ and any subsequence (x_{n_i}), one has that $\|\frac{1}{k} \sum_{i=1,k} \varepsilon_i T(x_{n_i})\|_F \geq \delta$. Follow [2, Section III, Theorem 1], with the obvious modifications, to prove that the property $\mathcal{ABS}$ for operators is equivalent to the negation, $(-\mathcal{P}_1)$, of the property $(\mathcal{P}_1)$. Then, proceed as in [2, Section II, Proposition 3], with the obvious modifications, to obtain that if $T \in \mathfrak{R}(E,F)$ then $T \in \mathcal{ABS}(E,F)$ if and only if $T \in \mathcal{BSR}(E,F)$, which proves $\mathfrak{R} \circ \mathcal{BSR}(E,F) \subset \mathcal{ABS}(E,F)$. Finally, note that $(-\mathcal{P}_1)$ implies the Rosenthal property and remember that $\mathcal{ABS}(E,F) \subset \mathcal{BSR}(E,F)$. ■

§3. Interpolation of Operators.

Let $\overline{A}$ and $\overline{B}$ be interpolation pairs of the Lions-Peetre real method of interpolation. If $T : \overline{A} \to \overline{B}$ is an interpolation operator, the induced operator from the intersection space $\mathcal{J}(\overline{A})$ into the sum space $\mathcal{S}(\overline{B})$ will be denoted by $T_{\mathcal{JS}}$.

Definition 3.1. An operator ideal $\mathfrak{I}$ satisfies the *Strong Property of Interpolation*, with respect to the real method of Lions-Peetre, abbreviated SPI, with the parameters $\theta \in (0,1)$ and $p \in (1,\infty)$ if the following occurs: the interpolated operator $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ belongs to $\mathfrak{I}$ if $T_{\mathcal{JS}} \in \mathfrak{I}(\mathcal{J}(\overline{A}), \mathcal{S}(\overline{B}))$.

The following theorem faithfully follows Heinrich's technique from the proof of [7, Proposition 2.2]:

Theorem 3.2. Any injective and surjective operator ideal satisfying the $\sum_p$-condition for some $p \in (1,\infty)$ has the SPI with parameters θ and p for all

$\theta \in (0,1)$.

Proof. Let $\overline{A}$ and $\overline{B}$ be interpolation pairs. In order to avoid a complicated notation, write A for the intersection $\mathcal{J}(\overline{A})$ and B for the sum $\mathcal{S}(\overline{B})$. Define on A and B the following norms, equivalent to the norms of intersection and sum spaces, respectively:

$$\|x\|_m = 2^{-\theta m} J(2^m, x) \quad \text{for } x \in A \text{ and } m \in \mathbb{Z},$$

$$\|y\|_m = 2^{-\theta m} K(2^m, y) \quad \text{for } y \in B \text{ and } m \in \mathbb{Z},$$

(see [3, Chapter 3] for the definition of norms J and K).

Denote by A_m the space $(A, \| \quad \|_m)$ and by B_m the space $(B, \| \quad \|_m)$. For each $(x_m)_{m\in\mathbb{Z}} \in (\sum_{m\in\mathbb{Z}} A_m)_p$, the sum $\sum_{m\in\mathbb{Z}} x_m$ converges (absolutely) in $\mathcal{S}(\overline{A})$. Then, there is a surjection Q from $(\sum_{m\in\mathbb{Z}} A_m)_p$ onto $\overline{A}_{\theta,p} = J_{\theta,p}(\overline{A})$:

$$Q(x_m)_{m\in\mathbb{Z}} = \sum_{m\in\mathbb{Z}} x_m \quad \text{(convergence in } \mathcal{S}(\overline{A})),$$

and an isomorphic embedding I from $\overline{B}_{\theta,p}$ into $(\sum_{m\in\mathbb{Z}} B_m)_p$ defined by $I(y) = (\ldots y, y, y, \ldots)$.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator and suppose that $T_{\mathcal{JS}} \in \mathfrak{I}(\mathcal{J}(\overline{A}), \mathcal{S}(\overline{B}))$. Denote by P_i the natural embedding of A_i into $(\sum_{m\in\mathbb{Z}} A_m)_p$ and by Q_j the natural projection of $(\sum_{m\in S} B_m)_p$ onto B_j. The operator $Q_j I T_{\theta,p} Q P_i$ is just $T_{\mathcal{JS}}$. Therefore, it is an operator of the class $\mathfrak{I}$ and, since $\mathfrak{I}$ satisfies the $\sum_p$-condition, the operator $IT_{\theta,p}Q$ belongs to $\mathfrak{I}((\sum_{m\in\mathbb{Z}} A_m)_p, (\sum_{m\in\mathbb{Z}} B_m)_p)$. Now, injectivity and surjectivity of $\mathfrak{I}$ imply that $T_{\theta,p} \in \mathfrak{I}(\overline{A}_{\theta,p}, \overline{B}_{\theta,p})$. ■

The proof of Lemma 3.3 below follows the guidelines of Heinrich from the proof of the respective theorem for BS, see [7, page 407].

Lemma 3.3. The operator ideal $\mathcal{BSR}$ satisfies the $\sum_p$-condition for any $p \in (1, \infty)$.

Proof. Use the following fact for the verification of the $\sum_p$-condition: the operator ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition provided the following holds for arbitrary Banach spaces E, F and G_m, $m \in \mathbb{Z}$: $U \in \mathfrak{L}(E, (\sum_{m\in\mathbb{Z}} G_m)_p), V \in \mathfrak{L}((\sum_{m\in\mathbb{Z}} G_m)_p), F)$ and $VJ_mQ_mU = VP_mU \in \mathfrak{I}(E, F)$ for all m, imply $VU \in \mathfrak{I}(E, F)$.

Assume that $VP_mU \in \mathcal{BSR}(E, F)$, for all m, and take a weakly null sequence (x_n) from E. Use the Erdös-Magidor result on regular methods of

summability (see [5]), a diagonal argument and proceed as does Heinrich, to obtain the conclusion that there exists a subsequence (x_{n_i}), for which the Cesàro averages $(\frac{1}{k}\sum_{i=1,k} VU(x_{n_i}))$ converge, which proves that $VU \in \mathcal{BSR}(E,F)$.■

The next theorem was obtained, partially, in [10]:

Theorem 3.4. The operator ideals $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}, \mathfrak{Q}$, their dual ideals $\mathfrak{X}^{dual}$, $\mathfrak{R}^{dual}$, $\mathcal{BS}^{dual}, \mathcal{ABS}^{dual}$, $\mathfrak{Q}^{dual}$ and mixed operator ideals as, for example, $\mathfrak{M} = \mathfrak{X} \cap \mathfrak{W}$, $\mathfrak{M} = \mathfrak{R} \cap \mathfrak{R}^{dual}$, $\mathfrak{M} = (\mathfrak{X} \cap \mathcal{ABS})^{dual}$ or $\mathfrak{M} = \mathfrak{X} \cap \mathfrak{R}^{dual} \cap \mathfrak{Q}$ are injective, surjective and satisfy the $\sum_p$-condition for any $p \in (1,\infty)$. Therefore, all of them satisfy the SPI for any $\theta \in (0,1)$ and $p \in (1,\infty)$.

Proof. It is well known the fact that $\mathfrak{W}$ and $\mathfrak{R}$ satisfy the condition. Heinrich proved that $\mathcal{BS}$ and $\mathfrak{Q}$ satisfy it, see [7] . Clearly, $\mathfrak{X}$ also satisfies it. For $\mathcal{ABS}$, use Lemma 3.3 and Theorem 2.3. Finally, remember that if each operator ideal satisfies the $\sum_p$-condition, for any $p \in (1,\infty)$, their dual ideals and their intersections also satisfy it for any $p \in (1,\infty)$. To conclude, apply Theorem 3.2. It is clear that the condition $p \in (1,\infty)$ is necessary. ■

For the case of $\mathcal{ABS}$, Theorem 3.4 provides a result due to A. Kryczka, [8, Corollary 4.2.].

See [11] for the generalization of these results to methods of interpolation between families of Banach spaces instead of pairs.

§4. Factoring Operators.

The operator ideal $\mathfrak{I}$ has the *factorization property* if for every operator $T \in \mathfrak{I}(E,F)$, there exist a Banach space $X \in \mathit{Space}(\mathfrak{I})$, and operators $U \in \mathfrak{L}(E,X), V \in \mathfrak{L}(X,F)$, in such a way that $T = VU$, see [7, Section II].

Several remarkable operator ideals have the factorization property, e.g., $\mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}$ and $\mathfrak{Q}$. For $\mathfrak{W}$, it was proved by Davis, Figiel, Johnson and Pełczyński in [4]; for $\mathfrak{R}$ by Beauzamy in [1, chapitre II, §3, Proposition 5]; for $\mathcal{BS}$ and $\mathcal{ABS}$ by Beauzamy in [2, Section IV, Theorems 1 and 2]; for $\mathfrak{Q}$ by Heinrich in [7, Theorem 2.1].

Other, also remarkable, operator ideals do not have the factorization property. For example, the operator ideal $\mathfrak{Y}$ does not have it, as was proved by N. Ghoussoub and W. B. Johnson in [6]. Neither does $\mathcal{BSR}$ have it, see [2].

The next lemma follows from the proof of Theorem 2.1 in [7], or from [1, page 37]:

Lemma 4.1. Any surjective operator ideal satisfying the SPI for some $p \in (1,\infty)$ and $\theta \in (0,1)$ has the factorization property.

This Lemma leads to the main result of this paper, see also [10]:

Corollary 4.2. All the operator ideals from Theorem 3.4 have the factorization property.

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Alexi Quevedo Suárez
Facultad de Ciencias
Escuela de Matemáticas
UCV, Caracas, Venezuela

alexi.quevedo@ciens.ucv.ve