FACTORIZATION OF MIXED OPERATORS

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In memory of my beloved Father, Bernardo,

Abstract

Let $T : E \to F$ be an operator between Banach spaces which, at the same time, is separable, Rosenthal and decomposing, for example. Using the real method of interpolation of Lions-Peetre for pairs, it is proved that there exists a Banach space S, which at the same time is separable, without copy of ℓ_1 and whose dual, S^*, possesses the Radon-Nikodym property, through which T factors. A technique to produce such factorization spaces for *mixed operators* is given. For this is necessary to prove, first, that many operator ideals possess the strong property of interpolation for the real method of Lions-Peetre.

Key words and phrases: Banach spaces, factorization of operators, real method of interpolation.

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1. Introduction. After 1974, when the famous and celebrated paper of Davis, Figiel, Johnson and Pełczyński on factorization of weakly compact operators, DFJP in short, was published, see [5], B. Beauzamy and S. Heinrich, using the real method of interpolation of Lions-Peetre, produced more results of this kind. They proved that the operator ideals of Rosenthal operators, Banach-Saks, alternate-sign Banach-Saks and decomposing have the factorization property. In this paper, mainly following a technique developed by Heinrich, see [8], it is shown that the ideals of separable, Rosenthal, weakly compact, Banach-Saks, alternate-signs Banach-Saks, decomposing, their dual ideals and many other ideals (*chains*), first, possess the strong property of interpolation respect the real method of interpolation and after, following a technique introduced by Beauzamy in [2], the factorization theorem for mixed operators is obtained.

A good, and easy example to understand what factorization of a mixed operator means, is one operator between Banach spaces which at the same time is weakly compact and separable: it was proved by W. B. Johnson, even before than DFJP, that such operators can be factored though a separable and reflexive space.

If $\mathfrak{I}$ is an operator ideal, if $\overline{A}$ and $\overline{B}$ are interpolation pairs then, the ideal $\mathfrak{I}$ possesses the *Interpolation Property* for a method $\mathcal{F}$ of interpolation (see [3], chap. 2) if the interpolated operator $T_{\mathcal{F}} : \mathcal{F}(\overline{A}) \to \mathcal{F}(\overline{B})$ belongs to $\mathfrak{I}$ when all (or some) of the extreme operators $T_i : A_i \to B_i$ $(i = 0, 1)$, belong to $\mathfrak{I}$. The ideal possesses the *Strong Property of Interpolation*, SPI in short, if the interpolated operator $T_{\mathcal{F}}$ belongs to $\mathfrak{I}$ when the induced $T_{\mathcal{J}\mathcal{S}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$ from the intersection into the sum spaces is in $\mathfrak{I}$.

The main result of this paper is achieved in §6, theorem 6.2, after a necessary preamble. In §5, theorem 5.4 it is proved the necessary result on interpolation.

Notation is standard; in any case, unexplained symbols, terms, concepts, etc., will be found in [3], [8] or [13].

2. Preliminaries. $\mathfrak{L}(E, F)$ is the space of all bounded linear operators. An operator ideal $\mathfrak{I}$ is a class of bounded linear operators such that the components $\mathfrak{I}(E, F) = \mathfrak{I} \bigcap \mathfrak{L}(E, F)$ satisfy the following conditions:(i) $\mathfrak{I}(E, F)$ is a linear subspace of $\mathfrak{L}(E, F)$, (ii) $\mathfrak{I}(E, F)$ contains the finite rank operators and (iii) if $R \in \mathfrak{L}(X, E), S \in \mathfrak{I}(E, F)$ and $T \in \mathfrak{L}(F, Y)$ then, $TSR \in \mathfrak{I}(X, Y)$ (see [8] and [13]).

The operator ideal is injective if for every isomorphic embedding $J \in \mathfrak{L}(F, Y)$ one has that $T \in \mathfrak{L}(E, F)$ and $JT \in \mathfrak{I}(E, Y)$ implies $T \in \mathfrak{I}(E, F)$; it is surjective if for every surjection $Q \in \mathfrak{L}(X, E)$ one has that $T \in \mathfrak{L}(E, F)$ and $TQ \in \mathfrak{I}(X, F)$ implies $T \in \mathfrak{I}(E, F)$. The ideal is closed if the components $\mathfrak{I}(E, F)$ are closed subspaces of $\mathfrak{L}(E, F)$ (see [8] and [13], chap. 4).

Every operator ideal $\mathfrak{I}$ defines a class of Banach spaces, *Space*$(\mathfrak{I})$, in the following way: $E \in$ *Space*$(\mathfrak{I})$ if and only if $1_E \in \mathfrak{I}(E, E)$.

If $(X_m)_{m\in\mathbb{Z}}$ is a family of Banach spaces, denote by $(\sum\limits_{m\in\mathbb{Z}} X_m)_p$, with $1 \leq p < \infty$, the space of all the maps $(x_m)_{m\in\mathbb{Z}}$, such that $x_m \in X_m$ with the norm $\|(x_m)_{m\in\mathbb{Z}}\| = (\sum\limits_{m\in\mathbb{Z}} \|x_m\|^p_{X_m})^{1/p} < \infty$. Denote by J_i the natural embedding of X_i into $(\sum\limits_{m\in\mathbb{Z}} X_m)_p$ and by Q_j the projection of $(\sum\limits_{m\in\mathbb{Z}} X_m)_p$ onto X_j.

Definition 2.1. The ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition for $1 \leq p < \infty$ (see [8]), if for any two families $(E_m)_{m\in\mathbb{Z}}$ and $(F_m)_{m\in\mathbb{Z}}$ of Banach spaces the following is true: if $T \in \mathfrak{L}((\sum\limits_{m\in\mathbb{Z}} E_m)_p, (\sum\limits_{m\in\mathbb{Z}} F_m)_p)$ and $Q_j T J_i \in \mathfrak{I}(E_i, F_j)$ for every $i, j \in \mathbb{Z}$, then, $T \in \mathfrak{I}((\sum\limits_{m\in\mathbb{Z}} E_m)_p, (\sum\limits_{m\in\mathbb{Z}} F_m)_p)$.

3. The Real Method of Interpolation. Let $\overline{X} = (X_0, X_1)$ be an interpolation pair of Banach spaces, that is, X_0 and X_1 are Banach spaces imbedded into a Hausdorff topological vector space $\mathcal{H}$. Denote by $\mathcal{J}(\overline{X})$ the intersection $X_0 \cap X_1$ and by $\mathcal{S}(\overline{X})$ the sum $X_0 + X_1$ with the norms

$$\|x\|_{\mathcal{J}(\overline{X})} = \max(\|x\|_{X_0}, \|x\|_{X_1})$$

and

$$\|x\|_{\mathcal{S}(\overline{X})} = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + \|x_1\|_{X_1}).$$

Let t be >0; define for $x \in \mathcal{S}(\overline{X})$ the K-functional

$$K(t,x) = K(t,x,\overline{X}) = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + t\|x_1\|_{X_1})$$

and, for $x \in \mathcal{J}(\overline{X})$, the J functional

$$J(t,x) = J(t,x,\overline{X}) = \max(\|x\|_{X_0}, t\|x\|_{X_1}).$$

For $0 < \theta < 1$ and $1 \leq p < \infty$ the space $K_{\theta,p}(\overline{X})$ is that of all $x \in \mathcal{S}(\overline{X})$ for which

$$\left[\sum_{m\in\mathbb{Z}} (2^{-\theta m} K(2^m, x))^p\right]^{1/p} < \infty.$$

The space $J_{\theta,p}(\overline{X})$ consists of those $x \in \mathcal{S}(\overline{X})$ for which there exists a map, $(x_m)_{m\in\mathbb{Z}}$, from $\mathbb{Z}$ into $\mathcal{J}(\overline{X})$, so that

$$x = \sum_{m\in\mathbb{Z}} x_m \qquad \text{(convergence in } \mathcal{S}(\overline{X}))$$

with

$$\left[\sum_{m\in\mathbb{Z}} (2^{-\theta m} J(2^m, x_m))^p\right]^{1/p} < \infty.$$

In each case, the norm is:

$$\|x\|_{\theta,p;K} = \left[\sum_{m\in\mathbb{Z}} (2^{-\theta m} K(2^m, x))^p\right]^{1/p}$$

and

$$\|x\|_{\theta,p;J} = \inf_{x=\Sigma x_m} \left[\sum_{m\in\mathbb{Z}} (2^{-\theta m} J(2^m, x_m))^p\right]^{1/p}.$$

The spaces $K_{\theta,p}(\overline{X})$ and $J_{\theta,p}(\overline{X})$ are interpolation spaces with respect to $\overline{X}$. They are, in fact, equal and their norms are equivalent ([3]). Accordingly, either $K_{\theta,p}(\overline{X})$ or $J_{\theta,p}(\overline{X})$ will be the space $\overline{X}_{\theta,p}$ of the real method of interpolation.

Now, let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two interpolation pairs. If $T : \mathcal{S}(\overline{A}) \to \mathcal{S}(\overline{B})$ is a bounded linear operator whose restriction to A_i is bounded from A_i into $B_i (i = 0, 1)$ then, one says that T is an interpolation operator from $\overline{A}$ into $\overline{B}$, and writes $T : \overline{A} \to \overline{B}$. In this case the interpolated operator $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}, (0 < \theta < 1$ and $1 \leq p < \infty)$, is bounded.

4. Some Classes of Operators.

4.1. An operator $T \in \mathfrak{L}(E, F)$ is compact if the image $T(B_E)$ of the unit ball of E is relatively compact in the norm topology of F. The operator is weakly compact if $T(B_E)$ is a relatively weakly compact set, or, using the Eberlein-Smulian Theorem, if and only if every sequence (Tx_n) with $x_n \in B_E$ admits a weakly convergent subsequence. The operator T is separable if $T(E)$ is a separable subspace of F or, equivalently, if $T(B_E)$ is a separable subset of F.

$T \in \mathfrak{L}(E, F)$ is a Rosenthal operator if for each $s \in \mathfrak{L}(\ell_1, E)$ the composition Ts is not an isomorphic embedding; T is unconditionally summing if for each $s \in \mathfrak{L}(c_0, E)$ the composition Ts is not an isomorphic embedding. In other words, T does not 'transport' copies of ℓ_1 or c_0, respectively. Using Rosenthal and Bessaga-Pełczyński theorems, it is easy to obtain the following characterization of these operators: T is Rosenthal if and only if every bounded sequence (x_n) of E possesses a subsequence (x_{n_k}) such that (Tx_{n_k}) is weak Cauchy, that is, if and only if $T(B_E)$ is weakly pre-compact. T is unconditionally summing if and only if for every sequence (x_n) of E which is unconditionally summable (i.e., for every sequence (x_n) such that $\sum_{n=1,\infty} \mid f(x_n) \mid < \infty$ for all $f \in E^*$) the sequence (Tx_n) is unconditionally summable in the norm topology of F, see [10].

$T \in \mathfrak{L}(E, F)$ has the Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n) such that (Tx'_n) is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum_{k=1,n} Tx'_k$, converges in F. The operator T has the alternate-signs Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n) such that $((-1)^n Tx'_n)$ is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum_{k=1,n} (-1)^k Tx'_k$, converges in F. The operator T has the Banach-Saks-Rosenthal property if any weakly null sequence (x_n) of E possesses a subsequence (x'_n) such that the sequence of the averages $n^{-1} \sum_{k=1,n} Tx'_k$, converges in F. See [1] for a thorough study of these operators, see also [9].

Let (Ω, μ) be a probability space. An operator $X \in \mathfrak{L}(L_1(\Omega, \mu), E)$ is right-decomposable if there exists a μ-measurable and E-valued kernel $x(\omega)$, with $\omega \in \Omega$, such that for all $f \in L_1(\Omega, \mu)$:

$$X(f) = \int_\Omega f(\omega)x(\omega)d\mu.$$

$T \in \mathfrak{L}(E, F)$ is a Radon-Nikodym operator if TX is right-decomposable for every $X \in (L_1(\Omega, \mu), E)$.

An operator $Z \in \mathfrak{L}(F, (L_\infty(\Omega, \mu))$ is left-decomposable if there exists a μ-measurable and F^*-valued kernel $z(\omega)$ such that for all $y \in F$:

$$Zy(\omega) = \langle y, z(\omega) \rangle.$$

$T \in \mathfrak{L}(E, F)$ is a decomposing operator if ZT is left decomposable for every $Z \in \mathfrak{L}(F, (L_\infty(\Omega, \mu))$; see [13], Chap. 24.

The following characterization of decomposing operators is well known, see [13]: T is decomposing if and only if its dual, T^*, is a Radon-Nikodym operator.

All the classes of operators defined above are ideals of operators in the sense of Pietsch. As in [13], capital gothic letters will denote each one of them. So: $\mathfrak{K}$ will be the ideal of compact operators, $\mathfrak{W}$ that of weakly compact, $\mathfrak{X}$ separable operators, $\mathfrak{R}$ Rosenthal operators, $\mathfrak{U}$ unconditionally summing, $\mathfrak{Y}$ Radon-Nikodym and $\mathfrak{Q}$ that of decomposing operators; see [13] for a detailed study of these ideals.

Place $\mathcal{BS}$ for Banach-Saks operator ideal, $\mathcal{ABS}$ for alternate-signs Banach-Saks and $\mathcal{BSR}$ for Banach-Saks-Rosenthal operator ideal. These last ideals are not treated by Pietsch. See [1] for their study.

All these ideals are closed and injective. Compact, weakly compact, separable, Rosenthal, Banach-Saks, alternate-signs Banach-Saks and decomposing are also surjective operator ideals. Neither $\mathfrak{U}$ nor $\mathfrak{Y}$ nor $\mathcal{BSR}$ are surjective.

A bounded subset A of the space X is called limited if $\lim\limits_{n\to\infty} \sup\limits_{x\in A} |x_n^*(x)| = 0$ for every $weak^*$-null sequence (x_n^*) in X^*, i.e., $\lim\limits_{n\to\infty} x_n^*(x) = 0$ uniformly on A. The operator $T \in \mathfrak{L}(E, F)$ is limited if $T(B_E)$ is a limited subset of F. Clearly, T is limited if and only if $T^* : F^* \to E^*$ takes $weak^*$-null sequences to norm null sequences, see [4].

4.2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two operator ideals. The product $\mathfrak{D} \circ \mathfrak{C}$ is a new operator ideal defined as follows: $T \in \mathfrak{L}(E, F)$ belongs to $\mathfrak{D} \circ \mathfrak{C}$ if there exists a Banach space G and operators $U \in \mathfrak{C}(E, G), V \in \mathfrak{D}(G, F)$, such that $T = VU$.

Heinrich, in [8], Thm. 1.1, proves that if $\mathfrak{C}$ and $\mathfrak{D}$ are closed then $\mathfrak{D} \circ \mathfrak{C}$ is also closed. Always is true that $\mathfrak{D} \circ \mathfrak{C} \subset \mathfrak{D} \cap \mathfrak{C}$ but the converse inclusion is not valid in general. Nevertheless, see [8], Thm. 1.3, if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective then $\mathfrak{D} \circ \mathfrak{C} = \mathfrak{D} \cap \mathfrak{C}$.

This product is, certainly, associative, non commutative in general, although, if $\mathfrak{C}$ and $\mathfrak{D}$ are closed, injective and surjective operator ideals, the product commutes. The identity element is, of course, $\mathfrak{L}$.

Also, is clear that if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective, both satisfying the $\sum_p$-condition, $1 \le p < \infty$, then, $\mathfrak{D} \circ \mathfrak{C} = \mathfrak{D} \cap \mathfrak{C}$ satisfies the $\sum_p$-condition.

Let $\mathfrak{I}$ be an operator ideal. The operator $T \in \mathfrak{L}(E, F)$ belongs to the dual ideal $\mathfrak{I}^{dual}$ if the adjoint operator T^* belongs to $\mathfrak{I}(F^*, E^*)$. For example, the fact that T is decomposing if and only if its dual, T^*, is a Radon-Nikodym operator, means that $\mathfrak{Q} = \mathfrak{Y}^{dual}$.

If $\mathfrak{I}$ is injective, $\mathfrak{I}^{dual}$ is surjective and if $\mathfrak{I}$ is surjective, $\mathfrak{I}^{dual}$ is injective (see [13], chap. 4). If $\mathfrak{I}$ is closed, so is $\mathfrak{I}^{dual}$. If $\mathfrak{I}$ satisfies the $\sum_p$-condition, for all

p with $1 < p < \infty$, then $\mathfrak{I}^{dual}$ satisfies $\sum_p$-condition for all p with $1 < p < \infty$.

Some times happens that, depending on the relationship between the ideals, the product reduces, as for example, the product $\mathfrak{W} \circ \mathcal{BS} \circ \mathfrak{R}$ which is equal to $\mathfrak{R}$, or, as in the case of weakly compact operators, for which $\mathfrak{W}^{dual} = \mathfrak{W}$, etc., (see [13]).

The product of several operator ideals will be called a *chain*. For example, $\mathfrak{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathfrak{I} = (\mathfrak{Q} \circ \mathcal{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, all them being, certainly, injective, surjective and closed operator ideals, are chains.

Let E and F be Banach spaces. An operator $T \in chain(E, F)$ is a *mixed operator*. For example, $T \in \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}(E, F)$, which means that $T : E \to F$ is <u>at the same time</u> a separable, dual Rosenthal and dual Radon-Nikodym operator, is a mixed operator of type $\mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$.

4.3. The relationships $\mathcal{BS} \subset \mathcal{ABS} \subset \mathcal{BSR}$, strict inclusions, are well known. The following theorem gives relationships much more precise that will be used in the next sections:

Theorem 4.4. *i)*$\mathcal{BS} = \mathfrak{W} \circ \mathcal{BSR}$, *ii)*$\mathcal{BS} = \mathfrak{W} \circ \mathcal{ABS}$ and *iii)*$\mathcal{ABS} = \mathfrak{R} \circ \mathcal{BSR}$ (the products must be in this order because $\mathcal{BSR}$ is not surjective).

Proof. *i)* for $\mathfrak{W} \circ \mathcal{BSR} \subset \mathcal{BS}$ use the Eberlein-Smulian characterization of weakly compact operators; for $\mathcal{BS} \subset \mathfrak{W} \circ \mathcal{BSR}$, use that $\mathcal{BS} \subset \mathfrak{W}$, (see [2], Lemma 1, page 49). Item *ii)* follows at once from *i)*. Item *iii)* is exactly the reformulation for operator ideals of the well known fact that for Banach spaces without copies of ℓ_1, $\mathcal{ABS}$ and $\mathcal{BSR}$ properties, are equivalent, (see [1], Section II, Prop. 3); the proof of *iii)* is the same given there, with the obvious modifications./

Now, the objective will be interpolate and factorize mixed operators.

5. Interpolation of Operators.

Interpolation theory is concerned with the following: let $\overline{A}$ and $\overline{B}$ be interpolation pairs and $T : \overline{A} \to \overline{B}$ an interpolation operator; if the extreme operators $T_i : A_i \to B_i$ $(i = 0, 1)$, or some of them, belong to a class $\mathfrak{I}$ of operators, what can be expected from the interpolated $T_{\theta,p}$? (see [14]).

If $T : \overline{A} \to \overline{B}$ is an interpolation operator, denote by $T_{\mathcal{JS}}$ the induced operator from $\mathcal{J}(\overline{A})$ into $\mathcal{S}(\overline{B})$.

Definition 5.1. An operator ideal $\mathfrak{I}$, possesses the *Strong Property of Interpolation* (SPI, in short), respect to the Real Method of interpolation, depending on the parameters $0 < \theta < 1$ and $1 \leq p < \infty$, if the following holds: the interpolated operator $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ belongs to $\mathfrak{I}$ if and only if $T_{\mathcal{JS}} \in \mathfrak{I}$.

Theorem 5.2. Any (closed), injective and surjective operator ideal, $\mathfrak{I}$, which satisfies the $\sum_p$-condition, possess SPI, see [14].

Proof. Let $\overline{A}$ and $\overline{B}$ be interpolation pairs. In order to avoid a complicated notation, write A for the intersection $\mathcal{J}(\overline{A})$ and B for the sum $\mathcal{S}(\overline{B})$. Define on

A and B the following equivalent norms (equivalent to the norms of intersection and sum spaces, respectively):

$$\|x\|_m = 2^{-\theta m} J(2^m, x) \quad \text{for } x \in A \text{ and } m \in \mathbb{Z},$$

$$\|y\|_m = 2^{-\theta m} K(2^m, y) \quad \text{for } y \in B \text{ and } m \in \mathbb{Z}.$$

Denote by A_m the space $(A, \| \ \|_m)$ and by B_m the space $(B, \| \ \|_m)$. For each $(x_m)_{m\in\mathbb{Z}} \in (\sum_{m\in\mathbb{Z}} A_m)_p$, the sum $\sum_{m\in\mathbb{Z}} x_m$ converges (absolutely) in $\mathcal{S}(\overline{A})$. Then, there is a surjection Q from $(\sum_{m\in\mathbb{Z}} A_m)_p$ onto $\overline{A}_{\theta,p} = J_{\theta,p}(\overline{A})$:

$$Q(x_m)_{m\in\mathbb{Z}} = \sum_{m\in\mathbb{Z}} x_m \quad (\text{convergence in } \mathcal{S}(\overline{A})),$$

and an isomorphic embedding J from $\overline{B}_{\theta,p}$ into $(\sum_{m\in\mathbb{Z}} B_m)_p$ defined by $J(y) = (\ldots y, y, y, \ldots)$.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator and assume that $T_{\mathcal{JS}} \in \mathfrak{I}$. Denote by J_i the embedding of A_i into $(\sum_{m\in\mathbb{Z}} A_m)_p$ and by Q_j the projection of $(\sum_{m\in S} B_m)_p$ onto B_j. The operator $Q_j JTQJ_i$ is just $T_{\mathcal{JS}}$. It is, then, an operator of the class $\mathfrak{I}$ and, since $\mathfrak{I}$ satisfies the $\sum_p$-condition, the operator JTQ belongs to $\mathfrak{I}((\sum_{m\in\mathbb{Z}} A_m)_p, (\sum_{m\in\mathbb{Z}} B_m)_p)$. Now, injectivity and surjectivity of $\mathfrak{I}$ imply that $T_{\theta,p} \in \mathfrak{I}(\overline{A}_{\theta,p}, \overline{B}_{\theta,p})$. Converse is clear./

Theorem 5.3. The single ideals $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BSR}, \mathcal{BS}, \mathcal{ABS}$ and $\mathfrak{Q}$ satisfy the $\sum_p$-condition for $1 < p < \infty$.

Proof. Let $\mathfrak{I}$ be one of these ideals. Prove first that if $(X_n)_{n\in\mathbb{N}}$ is a family of Banach spaces such that each X_n possesses the space property defined by $\mathfrak{I}$ then $(\sum_{n\in\mathbb{N}} X_n)_p, 1 < p < \infty$, also possesses this property. For $\mathcal{BS}$ and $\mathcal{WBS}$, this is a well known Theorem of J. R. Partington, see [12].

Let $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ be two families of Banach spaces, assume that $T \in \mathfrak{L}((\sum_{n\in\mathbb{N}} E_n)_p, (\sum_{n\in\mathbb{N}} F_n)_p), 1 < p < \infty$ and that $Q_j T J_i \in \mathfrak{I}(E_i, F_j)$ for every i, j; try to prove that $T \in \mathfrak{I}((\sum_{n\in\mathbb{N}} E_n)_p, (\sum_{n\in\mathbb{N}} F_n)_p)$ in two steps, for finite families first and, after, the general case:
(i) Easy for $\mathfrak{X}$, including the case $p = 1$.
(ii) For $\mathfrak{W}$ and $\mathfrak{R}$, use Eberlein-Smulian and Rosenthal Theorems. By a diagonal argument obtain, for all bounded sequence $[(x_n^k)_{n\in\mathbb{N}}]_{k\in\mathbb{N}}$ from $(\sum_{n\in\mathbb{N}} E_n)_p$,

a subsequence $[(x_n^{k_i})_{n\in\mathbb{N}}]_{i\in\mathbb{N}}$ such that $(T[(x_n^{k_i})_{n\in\mathbb{N}}])_{i\in\mathbb{N}}$ is weakly convergent or weakly Cauchy, respectively, in $(\sum_{n\in\mathbb{N}} F_n)_p$. Recall, for $1 < p < \infty : (\sum_{n\in\mathbb{N}} F_n)_p^* = (\sum_{n\in\mathbb{N}} F_n^*)_{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$.
(iii) For $\mathcal{BSR}$ (and $\mathcal{BS}$), the proof is that given by Heinrich in [8]. Use the following fact for the verification of the $\sum_p$-condition: the ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition if the following holds for arbitrary Banach spaces E, F and $G_n, (n = 1, 2, \ldots)$: $U \in \mathfrak{L}(E, (\sum_{n\in\mathbb{N}} G_n)_p), V \in \mathfrak{L}((\sum_{n\in\mathbb{N}} G_n)_p), F)$ and $VJ_nQ_nU = VP_nU \in \mathfrak{I}(E,F)$ for all n, implies $VU \in \mathfrak{I}(E,F)$.

Take a weakly null sequence (x_n) from E (respectively, a bounded sequence), use the Erdös-Magidor result on regular methods of summability (see [6]), a diagonal argument, and reason with Heinrich to conclude that there exists a subsequence (x'_n) which is Cesàro convergent, see [8], p. 407.
(iv) For $\mathcal{BS}$ and $\mathcal{ABS}$, use Theorem 4.4., recall that if each ideal satisfy the $\sum_p$-condition, the product also satisfies it and, finally, apply (ii) and (iii).
(v) For $\mathfrak{Q}$, see [8], p. 408.

Excepting the case of $\mathfrak{X}$, the assumption that $1 < p < \infty$ is, clearly, necessary./

Limited operators do not satisfy the $\sum_p$-condition for any p; they are like compact: according to the Josefson-Nissenzweig Theorem (see [4]), if the unit ball of a Banach space is a limited set, the space is finite dimensional.

Now is clear the next theorem. Except for $\mathcal{ABS}$, this result was obtained in [14]:

Theorem 5.4. The single ideals $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}, \mathfrak{Q}$, dual ideals $\mathfrak{X}^{dual}, \mathfrak{R}^{dual}$, $\mathcal{BS}^{dual}, \mathcal{ABS}^{dual}, \mathfrak{Q}^{dual}$ and chains as $\mathfrak{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathfrak{I} = (\mathfrak{X} \circ \mathcal{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, all possess the SPI.
proof. They are injective, surjective and satisfy $\sum_p$-condition for $1 < p < \infty$ (satisfying $\sum_p$-condition they are closed)./

Be clear, the chain $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, for example, possesses SPI, means that the interpolated $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ is, <u>at the same time</u>, a separable, dual Rosenthal and decomposing operator if and only if the induced $T_{\mathcal{JS}}$ is, <u>at the same time</u>, a separable, dual Rosenthal and decomposing operator.

For $\mathcal{ABS}$, it follows from Theorem 5.4., that result obtained by A. Kryczka, in [9], Corollary 4.2..

6. Factorization of Operators. The operator ideal $\mathfrak{I}$ possesses the factorization property if for every operator $T \in \mathfrak{I}$ there exists a Banach space $X \in Space(\mathfrak{I})$ and operators $U \in \mathfrak{L}(E, X), V \in \mathfrak{L}(X, F)$ such that $T = VU$.

Several very important operator ideals have the factorization property, e.g., $\mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}$ and $\mathfrak{Q}$. The fact that $\mathfrak{W}$ possesses the factorization property is, of course, the well known and celebrated factorization Theorem of Davis, Figiel, Johnson and Pełczyński in [5]. For $\mathfrak{R}, \mathcal{BS}$ and $\mathcal{ABS}$, it is a result of Beauzamy (see [1] and [2]) and for $\mathfrak{Q}$, it is due to Heinrich (see [8]).

Other, also important ideals, have not the factorization property. For example, neither Radon-Nikodym, $\mathfrak{Y}$, nor unconditionally summing, $\mathfrak{U}$, possess this property as was proved by Ghoussoub and Johnson in [7]. Limited operators do not possess the factorization property.

The following Lemma is of Beauzamy, [2]:

Lemma 6.1. Every surjective operator ideal with the SPI for the Lions-Peetre method for pairs (depending on $0 < \theta < 1$ and $1 < p < \infty$) possesses the factorization property.

proof. Let $T \in \mathfrak{L}(E, F)$ and denote by G the space $Im(T)$ with the norm obtained from the Minkowski functional of the convex, symmetric and absorbent set $T(B_E)$. This G is a Banach space isometrically isomorphic to the quotient space $E/KerT$ and is imbedded into F. By the surjectivity of the ideal, this imbedding $i : G \to F$ belongs to $\mathfrak{I}$. Since $\mathfrak{I}$ has the SPI, the space $(G, F)_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ belongs to $Space(\mathfrak{I})$ and T factors through $(G, F)_{\theta,p}$. Indeed, let $\overline{T}$ be the operator $\overline{T} : E \to G$; if u is the imbedding of G into $(G, F)_{\theta,p}$ and j that of $(G, F)_{\theta,p}$ into F, one has that T factors by $u\overline{T} : E \to (G, F)_{\theta,p}$ and $j : (G, F)_{\theta,p} \to F$. Take, for example, $(G, F)_{\frac{1}{2},2}$ as the factorization space./

This Lemma with Theorem 5.4 prove the following theorem:

Theorem 6.2. The single ideals $\mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}, \mathfrak{Q}$, dual ideals $\mathfrak{X}^{dual}, \mathfrak{R}^{dual}$, $\mathcal{BS}^{dual}, \mathcal{ABS}^{dual}, \mathfrak{Q}^{dual}$ and chains as $\mathfrak{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathfrak{I} = (\mathfrak{X} \circ \mathcal{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, are, all them, injective, surjective and possess the SPI, therefore, they possess the factorization property.

Remark 6.3. The ideal of limited operators does not possess SPI. In fact, being a surjective operator ideal, SPI would imply factorization property.

A few words about chains and factorization. An ideal such as $\mathfrak{R}^{dual}$, for example, possesses the factorization property, means, of course, that if $T \in \mathfrak{R}^{dual}(E, F)$ then T factors through a Banach space S, <u>whose dual</u>, S^*, has no isomorphic copy of ℓ_1. The chain $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$ possesses the factorization property, means that if $T \in \mathfrak{X} \circ \mathfrak{W}(E, F)$ then T factors through a Banach space $S \in space(\mathfrak{X} \circ \mathfrak{W})$, that is, a Banach space which, <u>at the same time</u>, is separable and reflexive. The chain $\mathfrak{I} = (\mathfrak{X} \circ \mathcal{ABS})^{dual}$ possesses the factorization property means that if $T \in (\mathfrak{X} \circ \mathcal{ABS})^{dual}(E, F)$ then T factors through a Banach space $S \in space((\mathfrak{X} \circ \mathcal{ABS})^{dual})$, that is, a space whose dual, S^*, is, at the same time, separable and possesses the alternate sign Banach-Saks property. Likewise for $T \in \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}(E, F)$.

The fact that $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$ possesses the factorization property is a result of W. B. Johnson obtained even before than DFJP, [5].

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