INTERPOLATING SEVERAL CLASSES OF OPERATORS

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Abstract. Many classes of operators have the strong interpolation property for the real method, that is, let $\mathcal{I}$ be one of the classes considered here, let $\bar{A}$, $\bar{B}$ be compatible pairs of Banach spaces, let $\bar{A}_{\theta,p}$, $\bar{B}_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ be the interpolation spaces obtained by the real method, $T : \bar{A} \to \bar{B}$ a bounded linear operator and T_{IS} the induced operator from the intersection $\mathsf{I}(\bar{A})$ into the sum $\mathsf{S}(\bar{B})$, then, the interpolated operator $T_{\theta,p}$ from $\bar{A}_{\theta,p}$ into $\bar{B}_{\theta,p}$ belongs to $\mathcal{I}$ if and only if $T_{\mathsf{IS}} \in \mathcal{I}$

Introduction.

Let $\bar{A}, \bar{B}$ be interpolation pairs and $\mathcal{I}$ be an operator ideal. One says that the ideal $\mathcal{I}$ has the interpolation property for a method $\mathcal{F}$ of interpolation (see [BL], chap. 2) if the interpolated operator $T_{\mathcal{F}} : \mathcal{F}\left(\bar{A}\right) \to \mathcal{F}\left(\bar{B}\right)$ belongs to $\mathcal{I}$ when the extreme operators $T_0 : A_0 \to B_0$ and (or) $T_1 : A_1 \to B_1$ belong to $\mathcal{I}$. The ideal possesses the strong interpolation property if the interpolated belongs to $\mathcal{I}$ when the induced $T_{\mathsf{IS}} : \mathsf{I}\left(\bar{A}\right) \to \mathsf{S}\left(\bar{B}\right)$ is in $\mathcal{I}$.

In this paper is showed that the ideals of separable, Rosenthal, weakly compact, Banach-Saks and Decomposing operators possess the strong interpolation property with respect to the real method of interpolation depending on the parameters $0 < \theta < 1$ and $1 < p < \infty$. Next, that the dual ideals also possess this strong property. Finally, following a technic of Beauzamy, some factorization theorems are obtained.

The notation is standard. $E, F, X, ...$ denote Banach spaces and $T, S, ...$ bounded linear operators. If X is a Banach space B_X is the closed unit ball of X and X^* is the dual space of X . By T^* one denotes the adjoint operator of T. In any case, unexplained terms or concepts will be find in [B], [BL], [H] or [P].

1. Preliminaries. If E and F are Banach spaces denote by $\mathcal{L}(E, F)$ the space of all bounded linear operators between E and F with the usual norm.

The space E is imbedded in F if E is an algebraic vector subspace of F and there exists a constant C such that $\|x\|_F \leq C\|x\|_E$ for all $x \in E$.

An operator ideal $\mathcal{I}$ is any subclass of the class $\mathcal{L}$ of all bounded linear operators between arbitrary Banach spaces such that the components $\mathcal{I}(E, F) = \mathcal{I} \cap \mathcal{L}(E, F)$ satisfy the following conditions: (i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$, (ii) $\mathcal{I}(E, F)$ contains the finite rank operators and (iii) if $R \in \mathcal{L}(X, E)$, $S \in \mathcal{I}(E, F)$ and $T \in \mathcal{L}(F, Y)$ then $TSR \in \mathcal{I}(X, Y)$; see [H] and [P].

The operator ideal $\mathcal{I}$ is injective if for every isomorphic embedding (monomorphism) $J \in \mathcal{L}(F, Y)$ one has that $T \in \mathcal{L}(E, F)$ and $JT \in \mathcal{I}(E, Y)$ imply $T \in \mathcal{I}(E, F)$; it is surjective if for every surjection $Q \in \mathcal{L}(X, E)$ one has that $T \in \mathcal{L}(E, F)$ and $TQ \in \mathcal{I}(X, F)$ imply $T \in \mathcal{I}(E, F)$. The ideal is closed if the components $\mathcal{I}(E, F)$ are closed subspaces of $\mathcal{L}(E, F)$ ([H] and [P], chap. 4).

Let $\mathcal{I}$ be an opertator ideal; the operator $T \in \mathcal{L}(E, F)$ belongs to the dual $\mathcal{I}^{dual}$ if the adjoint operator T^* is in $\mathcal{I}(F^*, E^*)$. Clearly $\mathcal{I}^{dual}$ is an operator ideal. If $\mathcal{I}$ is injective $\mathcal{I}^{dual}$ is surjective and if $\mathcal{I}$ is surjective $\mathcal{I}^{dual}$ is injective; if $\mathcal{I}$ is closed $\mathcal{I}^{dual}$ is closed ([P], chap. 4).

If $(X_m)_{m\in\mathbb{Z}}$ is a sequence of Banach spaces denote by $(\sum_{m\in\mathbb{Z}} \oplus X_m)_p$ with $1 \leq p < \infty$, the space of all sequences $(x_m)_{m\in\mathbb{Z}}$ with $x_m \in X_m$ and

$$\|(x_m)_{m\in\mathbb{Z}}\| = (\sum_{m\in\mathbb{Z}} \|x_m\|_{X_m}^p)^{1/p} < \infty.$$

Denote by J_i the natural (isometric) embedding of X_i into $(\sum_{m\in\mathbb{Z}} \oplus X_m)_p$ and by Q_j the projection of $(\sum_{m\in\mathbb{Z}} \oplus X_m)_p$ onto X_j.

The operator ideal $\mathcal{I}$ satisfies the $\sum_p$ condition (see [H]), if for any two sequences $(E_m)_{m\in\mathbb{Z}}$ and $(F_m)_{m\in\mathbb{Z}}$ of Banach spaces the following holds: if $T \in \mathcal{L}\left((\sum_{m\in\mathbb{Z}} \oplus E_m)_p, (\sum_{m\in\mathbb{Z}} \oplus F_m)_p\right)$ and $Q_j T J_i \in \mathcal{I}(E_i, F_j)$ for every $i, , j \in \mathbb{Z}$, then, $T \in \mathcal{I}\left((\sum_{m\in\mathbb{Z}} \oplus E_m)_p, (\sum_{m\in\mathbb{Z}} \oplus F_m)_p\right)$.

2. The Real Method of Interpolation. Let $\bar{X} = (X_0, X_1)$ be an interpolation pair of Banach spaces, that is, X_0 and X_1 are Banach spaces imbedded into a Hausdorff topological vector space $\mathcal{H}$. Denote by $\mathsf{I}(\bar{X})$ the intersection $X_0 \cap X_1$ and by $\mathsf{S}(\bar{X})$ the sum $X_0 + X_1$ with the norms

$$\|x\|_{\mathsf{I}(\bar{X})} = \max(\|x\|_{X_0}, \|x\|_{X_1})$$

and

$$\|x\|_{\mathsf{S}(\bar{X})} = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + \|x_1\|_{X_1}).$$

Let t be > 0; define for $x \in \mathsf{S}(\bar{X})$ the K-functional

$$K(t, x) = K(t, x, \bar{X}) = \inf_{x=x_0+x_1} (\|x_0\|_{X_0} + t\,\|x_1\|_{X_1})$$

and, for $x \in \mathsf{I}(\bar{X})$, the J functional

$$J(t, x) = J(t, x, \bar{X}) = \max\,(\|x\|_{X_0}, t\,\|x\|_{X_1}).$$

For $0 < \theta < 1$ and $1 \leq p < \infty$ the space $K_{\theta,p}(\bar{X})$ is that of all $x \in \mathsf{S}(\bar{X})$ for which

$$\left[\sum_{m\in\mathbb{Z}} \left(2^{-\theta m} K(2^m, x)\right)^p\right]^{1/p} < \infty.$$

The space $J_{\theta,p}(\bar{X})$ consists of those $x \in \mathsf{S}(\bar{X})$ for which there exists a sequence $(x_m)_{m\in\mathbb{Z}}$ of $\mathsf{I}(\bar{X})$ so that

$$x = \sum_{m\in\mathbb{Z}} x_m \qquad \text{(convergence in } \mathsf{S}(\bar{X})\text{)}$$

with

$$\left[\sum_{m\in\mathbb{Z}} \left(2^{-\theta m} J(2^m, x_m)\right)^p\right]^{1/p} < \infty.$$

In each case the norm is

$$\|x\|_{\theta,p;K} = \left[\sum_{m\in\mathbb{Z}} \left(2^{-\theta m} K(2^m, x)\right)^p\right]^{1/p}$$

and

$$\|x\|_{\theta,p;J} = \inf_{\sum x_m = x} \left[\sum_{m\in\mathbb{Z}} \left(2^{-\theta m} J(2^m, x_m)\right)^p\right]^{1/p}.$$

The spaces $K_{\theta,p}(\bar{X})$ and $J_{\theta,p}(\bar{X})$ are interpolation spaces with respect to $\bar{X}$. They are in fact equal and their norms equivalent ([BL]). Accordingly, either $K_{\theta,p}(\bar{X})$ or $J_{\theta,p}(\bar{X})$ will be the space $\bar{X}_{\theta,p}$ of the real method of interpolation.

Now, let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two interpolation pairs. If $T : \mathsf{S}(\bar{A}) \to \mathsf{S}(\bar{B})$ is a bounded linear operator whose restriction to A_i is bounded from A_i into B_i $(i = 0, 1)$ one says that T is a bounded operator from $\bar{A}$ into $\bar{B}$ and write $T : \bar{A} \to \bar{B}$. In this case the interpolated operator from $\bar{A}_{\theta,p}$ into $\bar{B}_{\theta,p}$ is bounded; it will be denoted by $T_{\theta,p}$.

3. Some Classes of Operators. An Operator $T \in \mathcal{L}(E, F)$ is compact if the image $T(B_E)$ of the unit ball of E is relatively compact in the norm topology of F; the operator is weakly compact if $T(B_E)$ is relatively weakly compact. T is a separable operator if $T(E)$ is a separable subspace of F or, equivalently, if $T(B_E)$ is a separable subset of F.

$T \in \mathcal{L}(E, F)$ is a Rosenthal operator if every bounded sequence (x_n) of E possesses a subsequence (x_{n_k}) such that (Tx_{n_k}) is weak Cauchy, that is, if $T(B_E)$ is weakly precompact. T is unconditionally summing if for every sequence (x_n) of E which is weakly unconditionally summable (i.e., for every sequence (x_n) such that $\sum\limits_{n=1,\infty} |f(x_n)| < \infty$ for all $f \in E^*$) $T(x_n)$ is unconditionally summable in the norm topology of F.

Using Rosenthal and Bessaga-Pelczynski theorems ([LT], § 2e) is easy to obtain the following characterizations of these operators :$T \in \mathcal{L}(E, F)$ is Rosenthal if and only if for each $s \in \mathcal{L}\ (l_1, E)$ the composition Ts is not an isomorphic embedding. T is unconditionally summing if and only if for every $s \in \mathcal{L}\ (c_0, E)$ the composition Ts is not an isomorphic embedding.

From the view point of the Operator Ideals Theory the class of Rosenthal operators is a remarkable one since it is the surjective hull of the ideal of strictly singular operators and the injective hull of that of strictly cosingular operators ([P], chap 4, § 4.7.21).

$T \in \mathcal{L}(E, F)$ is a Banach-Saks operator if any bounded sequence (x_n) of E has a subsequence (x_{n_k}) such that (Tx_{n_k}) is Césaro convergent ([B], page 49).

Let (Ω, μ) be a probability space. An operator $X \in \mathcal{L}(L_1(\Omega, \mu), E)$ is right decomposable if there exists a μ-measurable and E-valued kernel $x(\omega)$, with $\omega \in \Omega$, such that for all $f \in L_1(\Omega, \mu)$

$$Xf = \int\limits_{\Omega} f(\omega)x(\omega)d\mu.$$

$T \in \mathcal{L}(E, F)$ is called a Radon-Nikodym operator if TX is right decomposable for every $X \in \mathcal{L}(L_1(\Omega, \mu), E)$.

An operator $Z \in \mathcal{L}(F, L_\infty(\Omega, \mu))$ is left decomposable if there exists a μ-measurable and F^*-valued kernel $z(\omega)$ such that for all $y \in F$

$$Zy(\omega) = \langle y, z(\omega) \rangle.$$

$T \in \mathcal{L}(E, F)$ is a decomposing operator if ZT is left descomposable for every $Z \in \mathcal{L}(F, L_\infty(\Omega, \mu))$; see [P], chap. 24.

All the classes defined here are closed and injective operator ideals. Compact, weakly compact, separable, Rosenthal, Banach-Saks and decomposing are also surjective ideals. Unconditionally summing and Radon-Nikodym are not surjective.

As in [P], a special capital letter will denote each one of these ideals. So, it will be used $\mathcal{K}$ for the ideal of compact operators, $\mathcal{W}$ for weakly compact, $\mathcal{X}$ for separable, $\mathcal{R}$ for Rosenthal, $\mathcal{U}$ for unconditionally summing, $\mathcal{S}$ for Banach-Saks, $\mathcal{N}$ for Radon-Nikodym and $\mathcal{Q}$ for the ideal of decomposing operators.

Let T be an operator, $T \in \mathcal{L}(E,F)$;one says that T is a Tauberian operator if $(T^{**})^{-1}(F) = E$.

These operators are, in some sense, opposite to weakly compact since $T \in \mathcal{L}(E,F)$ is weakly compact if and only if $T^{**}(E^{**}) \subset F$ and, thus, if T is weakly compact and Tauberian E has to be reflexive.

Tauberian operators have been studied deeply by many authors. It must be mentioned specially the works of Kalton and Wilansky [KW], that of Neidinger-Rosenthal and the Ph.D. Dissertation of Neidinger [N_1].

The following properties of Tauberian injections will be used in the sequel. Their proof can be found in [N_1], chap.I, prop. 9.E (page 75), in [N_2], thm., 1.4 and in [KW].

Theorem 3.1. Let $j \in \mathcal{L}(E,F)$ be a Tauberian injection

(i) if B is a bounded set of E then, $j(B)$ is separable if and only if B is separable.

(ii) if the space E has no reflexive infinite dimensional subspaces, such as $E = c_0$ or $E = l_1$ then, j is an isomorphic embedding.

(iii) if B is a bounded set of E then, $j(B)$ is weakly compact if and only if B is weakly compact.

It seems that Tauberian operators appeared for the first time, as a differentiated class, in [GW] but is after 1974 when Davis, Figiel, Johnson and Pelczynski presented their famous factorization theorem that the abstract theory of this operators began to be developed.

By renorming $\bar{X}_{\theta,p}$ with an equivalent norm, inspired by [DFJP], B. Beauzamy stablished that for the Lions-Peetre interpolation spaces the imbedding of $\bar{X}_{\theta,p}$ in the sum $\mathsf{S}\left(\bar{X}\right)$ is a Tauberian injection for $0 < \theta < 1$ and $1 < p < \infty$ ([B], chap. 2,§2, prop. 1).

4. Interpolation of Operators. Interpolation theory is concerned with the following: let $\bar{A}$ and $\bar{B}$ be interpolation pairs and $T : \bar{A} \to \bar{B}$ bounded. If the extreme operators $T_i : A_i \to B_i$ for $i = 0, 1$ are, both or one of them, in a class $\mathcal{I}$ of operators what can be said about the interpolated $T_{\theta,p}$.

In the early 1960s, Lions and Peetre, studied this for the class of compact operators and stated that if one of the extreme operators T_0 or T_1 is compact then, the interpolated $T_{\theta,p}$ is also compact provided either A_0 and A_1 or B_0 and B_1 are equal (see [BL], thm. 3.8.1). Later, A. Persson by assuming certain aproximation conditions on B_0 and B_1 generalized their result to the case where $A_0 \neq A_1$ and $B_0 \neq B_1$ were allowed. In 1969 K. Hayakawa removed these conditions and showed that the interpolated operator $T_{\theta,p}$ is compact if T_0 and T_1, both, were compact. More recently, M. Cwikel in [C] proved that if only one of the extreme operators is compact the interpolated $T_{\theta,p}$ is compact for $\theta \in (0,1)$ and $1 \leq p \leq \infty$ with no extra assumptions.

S. Heinrich in [H], propositions 1.6 and 1.7 generalizes the aforesaid theorem of Lions-Peetre in the following way,

Theorem 4.1 [H]

(i) Let $\mathcal{I}$ be an injective and closed operator ideal, let A, B_0, B_1 be Banach spaces and suppose that (B_0, B_1) is an interpolation pair. If $T \in \mathcal{L}(A, B_0)$ and $T \in \mathcal{I}(A, B_1)$ then $T \in \mathcal{I}\left(A, \bar{B}_{\theta,p}\right)$ with $0 < \theta < 1$ and $1 \leq p \leq \infty$.

(ii) Let $\mathcal{I}$ be a surjective and closed operator ideal, let A_0, A_1, B be Banach spaces and suppose that (A_0, A_1) is an interpolation pair. If $T \in \mathcal{I}(A_0, B)$ and $T \in \mathcal{L}(A_1, B)$ then $T \in \mathcal{I}\left(\bar{A}_{\theta,p}, B\right)$ with $0 < \theta < 1$ and $1 \leq p \leq \infty$.

Some Banach space properties are inherited by the interpolation spaces $\bar{X}_{\theta,p}$. For example, H.Morimoto (cited in [B]) proved that if one of the spaces X_0 or X_1 is reflexive then $\bar{X}_{\theta,p}$ also is.

Beauzamy in [B] improved this result

Theorem 4.2 [B]. The spaces $\bar{X}_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ are reflexive if (and only if) the imbedding i from $\mathsf{I}(\bar{X})$ into $\mathsf{S}\left(\bar{X}\right)$ is weakly compact.

Moreover, he also obtained similar results for other Banach space properties

Theorem 4.3 [B]. Let $0 < \theta < 1$ and $1 < p < \infty$; let i be the imbedding from $\mathsf{I}\left(\bar{X}\right)$ into $\mathsf{S}\left(\bar{X}\right)$. Then,

(i) $\bar{X}_{\theta,p}$ is separable if (and only if) i is a separable operator.

(ii) $\bar{X}_{\theta,p}$ has not isomorphic copy of l_1 if (and only if) i is a Rosenthal operator.

Now, suppose that $\bar{A}, \bar{B}$ are interpolation pairs, let $T : \bar{A} \to \bar{B}$ be bounded and denote by T_{IS} the operator from $\mathsf{I}\left(\bar{A}\right)$ into $\mathsf{S}\left(\bar{B}\right)$ induced by T. An operator class has the strong interpolation property with respect to the real method, depending on the parameters $0 < \theta < 1$ and $1 < p < \infty$, if the interpolated operator $T_{\theta,p}$ is in that class when T_{IS} is.

Many operator ideals have the strong interpolation property. This will be proved in the next paragraph for the ideals $\mathcal{X}$, $\mathcal{R}$ and $\mathcal{W}$.

5. Interpolation of separable, Rosenthal and weakly compact operators. The aim of this paragraph is to prove the following theorem

Theorem 5.1. Let $\bar{A}, \bar{B}$ the interpolation pairs and suppose $T : \bar{A} \to \bar{B}$. Then, for $0 < \theta < 1$ and $1 < p < \infty$

(i) $T_{\theta,p}$ is separable if (and only if) T_{IS} is separable.

(ii) $T_{\theta,p}$ is Rosenthal if (and only if) T_{IS} is Rosenthal.

(iii) $T_{\theta,p}$ is weakly compact if (and only if) T_{IS} is weakly compact.

The proof will be an easy consequence of two lemmas.

Lemma 5.2. Let (X_0, X_1) be an interpolation pair and Y a Banach space; let $T : \mathsf{S}\left(\bar{X}\right) \to Y$ be a continuous operator. Then, for $0 < \theta < 1$ and $1 \leq p \leq \infty$

(i) $T : \bar{X}_{\theta,p} \to Y$ is separable if (and only if) $T : \mathsf{I}(\bar{X}) \to Y$ is separable.

(ii) $T : \bar{X}_{\theta,p} \to Y$ is Rosenthal if (and only if) $T : \mathsf{I}(\bar{X}) \to Y$ is Rosenthal.

(iii) $T : \bar{X}_{\theta,p} \to Y$ is weakly compact if (and only if) $T : \mathsf{I}(\bar{X}) \to Y$ is weakly compact.

Proof. Suppose that $T : \mathsf{I}(\bar{X}) \to Y$ is separable, Rosenthal or weakly compact. Since $\left(\mathsf{I}\left(\bar{X}\right), \mathsf{S}\left(\bar{X}\right)\right)$ is an interpolation pair and the classes of separable, Rosenthal and weakly compact operators are surjective closed operator ideals one has, by Theorem 4.1.ii that $T : \left(\mathsf{I}\left(\bar{X}\right), \mathsf{S}\left(\bar{X}\right)\right)_{\theta,p} \to Y$ with $0 < \theta < 1$ and $1 \leq p \leq \infty$ is separable, Rosenthal or weakly compact. Now, for $\frac{1}{2} \leq \theta < 1$ and $1 \leq p \leq \infty$ the well known identity

$$\left(\mathsf{I}\left(\bar{X}\right), \mathsf{S}\left(\bar{X}\right)\right)_{\theta,p} = \bar{X}_{\theta,p} + \bar{X}_{\theta,p} = \left(\mathsf{S}\left(\bar{X}\right), \mathsf{I}\left(\bar{X}\right)\right)_{1-\theta,p},$$

implies that $\bar{X}_{\theta,p}$ and $\bar{X}_{1-\theta,p}$ are imbedded in $\left(\mathsf{I}\left(\bar{X}\right), \mathsf{S}\left(\bar{X}\right)\right)_{\theta,p}$. Then, $T : \bar{X}_{\theta,p} \to Y$ whit $0 < \theta < 1$ and $1 \leq p \leq \infty$ is separable, Rosenthal or weakly compact.

The converse implications are obvious in all cases.

Remark 5.3. The proof of Lemma 5.2 could be done based on the cocept of almost absorbing developed by Neidinger in [N]:

Let W and V be subsets of the Banach space Y. The set W almost absorbs V if for every $\varepsilon > 0$ there exists $t > 0$ such that $V \subset tW + \varepsilon B_X$.

Using the fact that if $T : \mathsf{I}(\bar{X}) \to Y$ is continuous then $T(B_{\mathsf{I}(\bar{X})})$ almost absorbs $T(B_{\bar{X}_{\theta,p}})$ and the following proposition, a new proof of Lemma 5.2 can be obtained.

Proposition [N]. Let W and V be subsets of the Banach space Y. If W almost absorbs V and W is separable, weakly precompact or relatively weakly compact then, V is separable, weakly precompact or relatively weakly compact (see [N_1], [N_2] Thm.1.3 and [N_3] Lemma 3).

Lemma 5.4. Let (Y_0, Y_1) be an interpolation pair and X a Banach space. Let $1 < p < \infty$ and $T : X \to \bar{Y}_{\theta,p}$ a continuous operator. Then,

(i) $T : X \to \mathsf{S}(\bar{Y})$ is separable (if and) only if $T : X \to \bar{Y}_{\theta,p}$ is separable.

(ii) $T : X \to \mathsf{S}(\bar{Y})$ is Rosenthal (if and) only if $T : X \to \bar{Y}_{\theta,p}$ is Rosenthal.

(iii) $T : X \to \mathsf{S}(\bar{Y})$ is weakly compact (if and) only if $T : X \to \bar{Y}_{\theta,p}$ is weakly compact.

Proof.(i) Suppose that $T : X \to \mathsf{S}(\bar{Y})$ is separable. Since the imbeddding $J_{\theta,p} : \bar{Y}_{\theta,p} \to \mathsf{S}(\bar{Y})$ is a Tauberian injection and $J_{\theta,p}T(B_X)$ is separable one has by theorem 3.1.i that $T(B_X)$ is separable.

(ii) If $T : X \to \bar{Y}_{\theta,p}$ is not Rosenthal there exists $s \in \mathcal{L}(l_1, X)$ such that $Ts : l_1 \to \bar{Y}_{\theta,p}$ is an isomorphic embedding. Being $J_{\theta,p}$ a Tauberian injection its restriction to an isomorphic copy of l_1 is an isomorphic embedding by theorem 3.1.ii. So, $J_{\theta,p}Ts$ is an isomorphic embedding and $T : X \to \mathsf{S}(\bar{Y})$ is not Rosenthal.

(iii) Suppose that $T : X \to \mathsf{S}(\bar{Y})$ is weakly compact. Then, $J_{\theta,p}T(B_X)$ is relatively weakly compact and $T(B_X)$ is relatively weakly compact in $\bar{Y}_{\theta,p}$ by Thm. 3.1.iii. So, $T : X \to \bar{Y}_{\theta,p}$ is weakly compact.

Remark 5.5. Neidinger in [N] proved that if $J \in \mathcal{L}(E, F)$ is a Tauberian injection and B is a bounded set of E then $J(B)$ is weakly precompact if and only if B is weakly precompact. This fact provides a second proof of Lemma 5.4.ii.

A different proof of this Lemma 5.4.ii is suggested by a well known theorem of M. Levy [L]: every closed subspace of $\bar{Y}_{\theta,p}$ with $0 < \theta < 1$ and $1 \leq p \leq \infty$ which is not closed in $\mathsf{S}(\bar{Y})$ contains a (complemented) isomorphic copy of l_p. If $\bar{Y}_{\theta,p}$, with $1 < p < \infty$, contains a subspace isomorphic to l_1 this subspace has to be closed in $\mathsf{S}(\bar{Y})$ since l_1 has no copy of l_p with $p \neq 1$. Consequently, if $T : X \to \bar{Y}_{\theta.p}$ is not Rosenthal then, $T : X \to \mathsf{S}(\bar{Y})$ is not Rosenthal (see [L_2], pag. 17).

Now the proof of Theorem 5.1: if $T_{\mathsf{IS}} : \mathsf{I}(\bar{A}) \to \mathsf{S}(\bar{B})$ is separable, Rosenthal or weakly compact, $T : \bar{A}_{\theta,p} \to \mathsf{S}(\bar{B})$ is, by Lemma 5.2 separable, Rosenthal or weakly compact (for $1 \le p \le \infty$) and, by Lemma 5.4, $T_{\theta,p} : \bar{A}_{\theta,p} \to \bar{B}_{\theta,p}$ is separable, Rosenthal or weakly compact (for $1 < p < \infty$).

As inmediate corollaries of Theorem 5.1 one has Theorems 4.2 and 4.3. Also the following

Corolary 5.6. Let $\bar{A}, \bar{B}$ be interpolation pairs and $T : \bar{A} \to \bar{B}$. If one of the extreme operators $T_0 : A_0 \to B_0$ or $T_1 : A_1 \to B_1$ is separable, Rosenthal or weakly compact then, the interpolated operator $T_{\theta,p} : \bar{A}_{\theta,p} \to \bar{B}_{\theta,p}$ is separable, Rosenthal or weakly compact (for $0 < \theta < 1$ and $1 < p < \infty$).

Theorem 5.1 is not new. It was obtained also by F. Räbiger in [R], Thms. 8.2, 8.3 and 10.10 for a more general method of interpolation. For the classes $\mathcal{W}$ and $\mathcal{R}$ it was obtained in [MQ]. Moreover, related results are shown in [M], Thm.3.3 and [N$_3$], Thm. 5. It is stated here for completeness and in order to present the technic used in the proof, different to that of the next section.

6. Interpolation of Banach-Saks and Decomposing Operators. In this paragraph a different technic, inspired in the work [H], is introduced to prove that the ideals of Banach-Saks and Decomposing operators also have the strong interpolation property.

Let $\bar{A}, \bar{B}$ be interpolation pairs. In order to avoid a complicated notation write A for the intersection $\mathsf{I}(\bar{A})$ and B for the sum $\mathsf{S}(\bar{B})$. Define on A and B the following equivalent norms

$$\begin{aligned} \|x\|_m &= 2^{-\theta m} J\left(2^m, x\right) \qquad \text{for } x \in A \text{ and } m \in \mathbb{Z} \\ \|y\|_m &= 2^{-\theta m} K\left(2^m, x\right) \qquad \text{for } y \in B \text{ and } m \in \mathbb{Z}. \end{aligned}$$

Denote by A_m the space $(A, \| \quad \|_m)$ and by B_m the space $(B, \| \quad \|_m)$. For each $(x_m)_{m\in\mathbb{Z}} \in \left(\sum_{m\in\mathbb{Z}} \oplus A_m\right)_p$ the sum $\sum_{m\in\mathbb{Z}} x_m$ converges in $\mathsf{S}(\bar{A})$. Then, there is a surjection Q from $\left(\sum_{m\in\mathbb{Z}} \oplus A_m\right)_p$ onto $\bar{A}_{\theta,p} = J_{\theta,p}\left(\bar{A}\right)$ with $0 < \theta < 1$ and $1 < p < \infty$

$$Q\left((x_m)_{m\in\mathbb{Z}}\right)=\sum_{m\in\mathbb{Z}} x_m \qquad \text{(convergence in } \mathsf{S}\left(\bar{A}\right))$$

and an isomorphic embedding J from $\bar{B}_{\theta,p}=K_{\theta,p}\left(\bar{B}\right)$ into $\left(\sum_{m\in\mathbb{Z}}\oplus B_m\right)_p$ defined by $J(y)=(\ldots,y,y,y,\ldots)$.

Proposition 6.1. Let $0<\theta<1$ and $1<p<\infty$; let $\mathcal{I}$ be a (closed) injective and surjective operator ideal which satisfies the $\sum_p$ condition. Suppose that $\bar{A},\bar{B}$ are interpolation pairs and $T:\bar{A}\to\bar{B}$; let T_{IS} be the induced operator from $\mathsf{I}(\bar{A})$ into $\mathsf{S}(\bar{B})$. If $T_{\mathsf{IS}}\in\mathcal{I}(A,B)$ then the interpolated $T_{\theta,p}\in\mathcal{I}\left(\bar{A}_{\theta,p},\bar{B}_{\theta,p}\right)$.

Proof. Let J_i be the embedding of A_i into $\left(\sum_{m\in\mathbb{Z}}\oplus A_m\right)_p$ and Q_j the projection of $\left(\sum_{m\in\mathbb{Z}}\oplus B_m\right)_p$ onto B_j. Obviously the operator Q_jJTQJ_i is the operator T_{IS} from $A_i=\mathsf{I}\left(\overline{A}\right)$ into $B_j=\mathsf{S}\left(\bar{B}\right)$. It is, then, an operator of the class $\mathcal{I}$. Since $\mathcal{I}$ satisfies the $\sum_p$ condition the operator JTQ belongs to $\mathcal{I}\left(\left(\sum_{m\in\mathbb{Z}}\oplus A_m\right)_p,\left(\sum_{m\in\mathbb{Z}}\oplus B_m\right)_p\right)$. Now, the injectivity and surjectivity of $\mathcal{I}$ implies that $T:\bar{A}_{\theta,p}\to\bar{B}_{\theta,p}$ is in $\mathcal{I}$, that is, $T_{\theta,p}\in\mathcal{I}\left(\bar{A}_{\theta,p},\bar{B}_{\theta,p}\right)$.

It follows the main theorem of this section.

Theorem 6.2. The ideals $\mathcal{S}$ and $\mathcal{Q}$ possess the strong property of interpolation for the real method (depending on $0<\theta<1$ and $1<p<\infty$); that is, if $\bar{A},\bar{B}$ are interpolation pairs and $T:\bar{A}\to\bar{B}$ then, for $0<\theta<1$ and $1<p<\infty$

(i) $T_{\theta,p}$ is Banach-Saks if (and only if) T_{IS} is Banach-Saks.

(ii) $T_{\theta,p}$ is Decomposing if (and only if) T_{IS} is Decomposing.

Proof The ideals $\mathcal{S}$ and $\mathcal{Q}$ satisfy the $\sum_p$ condition, see [H, pages 407 to 409]; since they are injective and surjective, Proposition 6.1 aplies.

Corollary 6.3 [H]. Let $\bar{X}$ be an interpolation pair. The space $\bar{X}_{\theta,p}$ with $0<\theta<1$ and $1<p<\infty$, possesses the Banach-Saks property or its dual has the Radon-Nikodym property if and only if the imbedding i from $\mathsf{I}(\bar{X})$ into $\mathsf{S}\left(\bar{X}\right)$ is a Banach-Saks or Decomposing operator.

Corollary 6.4. The ideals $\mathcal{S}$ and $\mathcal{Q}$ possess the interpolation property, that is, if $\bar{A}, \bar{B}$ are interpolation pairs, $T : \bar{A} \to \bar{B}$ and one of the extreme operators T_0 or T_1 is Banach-Saks or Decomposing then the interpolated operator $T_{\theta,p}$, with $0 < \theta < 1$ and $1 < p < \infty$, is Banach-Saks or Decomposing.

Remark 6.5. Theorem 6.2 is not true neither for $p = 1$ nor for $p = \infty$. Indeed, according to M. Levy the interpolation spaces $\bar{X}_{\theta,1}$ and $\bar{X}_{\theta,\infty}$ contain, in the non trivial case, (complemented) isomorphic copies of l_1 and l_∞ respectively. Consequently, they have not the Banach-Saks property nor have dual with the Radon-Nikodym property since these properties are inherited by closed subspaces (see [L_2], chap. II, Theorem 2).

7. Interpolation of Dual Ideals. Here, the classes $\mathcal{X}^{dual}, \mathcal{R}^{dual}, \mathcal{S}^{dual}$ and $\mathcal{Q}^{dual}$ are studied from the view point of interpolation. As has been said in §1, they are closed, injective and surjective operator ideals.

Since $\mathcal{S} \subset \mathcal{W}$, is ovbious that the class of Banach-Saks is a subclass of $\mathcal{R}$, $\mathcal{U}$, $\mathcal{Q}$, $\mathcal{N}$ and of $\mathcal{R}^{dual}$, $\mathcal{U}^{dual}$, $\mathcal{Q}^{dual}$. Other order relations, not stated in [P], are collected in the next proposition; they are easy extensions of well known facts in Banach space theory (see [LT], prop.2.e.8).

Proposition 7.1.(i)$\mathcal{R} \subset \mathcal{U}^{dual}$, (ii) $\mathcal{X}^{dual} \subset \mathcal{U}^{dual}$ and (iii) $\mathcal{R}^{dual} \subset \mathcal{U}$. All inclusions are strict.

Let (X_0, X_1) be an interpolation pair; denote by $\bar{X}_0$ and $\bar{X}_1$ the closure of $\mathsf{I}\left(\bar{X}\right)$ in X_0 and X_1. It is well known that for the real method of interpolation one has for $0 < \theta < 1$ and $1 \le p < \infty$

$$\left(\bar{X}_0, X_1\right)_{\theta,p} = \left(\bar{X}_0, \bar{X}_1\right)_{\theta,p} = \left(X_0, \bar{X}_1\right)_{\theta,p}$$

and that $\left(\bar{X}_0, \bar{X}_1\right)_{\theta,p} = (X_0, X_1)_{\theta,p}$ (see [B] and [BL]). If $\mathsf{I}\left(\bar{X}\right)$ is dense in X_0 and X_1 it is also dense in $\mathsf{S}\left(\bar{X}\right)$; it can be proved, in this case, that

$$\mathsf{I}\left(\bar{X}\right)^* = X_0^* + X_1^* = \mathsf{S}\left(\bar{X}^*\right),$$
$$\mathsf{S}\left(\bar{X}\right)^* = X_0^* \cap X_1^* = \mathsf{I}\left(\bar{X}^*\right)$$

and, by the duality theorem, for $1 \le p < \infty$ that

$$\bar{X}^*_{\theta,p} = (X^*_0, X^*_1)_{\theta,q} \text{ (equivalent norms)},$$

with $\frac{1}{p} + \frac{1}{q} = 1$ (see [BL], chap. 2).

Lemma 7.2. Let $\mathcal{I}$ be a surjective operator ideal with the strong interpolation property for the real method (depending on $0 < \theta < 1$ and $1 < p < \infty$). Then, $\mathcal{I}^{dual}$ also possesses the strong interpolation property.

Proof. Let $\bar{A}, \bar{B}$ are interpolation pairs and assume that $T : \bar{A} \to \bar{B}$. One can soppose that $\mathsf{I}\left(\bar{A}\right)$ is dense in A_i and that $\mathsf{I}\left(\bar{B}\right)$ is dense in B_i for $i = 0, 1$ because $\bar{B}_0 + \bar{B}_1$ is a closed subspace of $\mathsf{S}\left(\bar{B}\right) = B_0 + B_1$ and, by the injectivity, T_{IS} is in $\mathcal{I}^{dual}$ if and only if $T : \mathsf{I}\left(\bar{A}\right) \to \bar{B}_0 + \bar{B}_1$ is in $\mathcal{I}^{dual}$. If T_{IS} is in $\mathcal{I}^{dual}$, that is, if $T^* : \mathsf{I}\left(\bar{B}^*\right) \to \mathsf{S}\left(\bar{A}^*\right)$ is in $\mathcal{I}$ then, by the duality theorem $T^*_{\theta,p} : \bar{B}^*_{\theta,p} \to \bar{A}^*_{\theta,p}$ is in $\mathcal{I}$ for $1 < p < \infty$ and $T_{\theta,p} : \bar{B}_{\theta,p} \to \bar{A}_{\theta,p}$ is in $\mathcal{I}^{dual}$. This proves the non trivial part of the Lemma.

From this one has inmediatly

Theorem 7.3. Let $\bar{A}, \bar{B}$ be interpolation pairs and $T : \bar{A} \to \bar{B}$. Then, for $0 < \theta < 1$ and $1 < p < \infty$

(i) $T_{\theta,p}$ is dual separable if and only if T_{IS} is dual separable.
(ii) $T_{\theta,p}$ is dual Rosenthal if and only if T_{IS} is dual Rosenthal.
(iii) $T_{\theta,p}$ is dual Banach-Saks if and only if T_{IS} is dual Banach-Saks.
(iv) $T_{\theta,p}$ is dual Decomposing if and only if T_{IS} is dual Decomposing.

Corollary 7.4. Let $\bar{X}$ be an interpolation pair. Then, the space $\bar{X}^*_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ is separable, has no isomorphic copy of l_1, possesses the Banach-Saks property or its dual has the Radon-Nikodym property if and only if the imbedding i from $\mathsf{I}(\bar{X})$ into $\mathsf{S}\left(\bar{X}\right)$ belongs to $\mathcal{X}^{dual}, \mathcal{R}^{dual}, \mathcal{S}^{dual}$ or $\mathcal{Q}^{dual}$ respectively.

Corollary 7.5. Let $\bar{A}$ and $\bar{B}$ be interpolation pairs and assume $T : \bar{A} \to \bar{B}$. If one the extreme operators T_0 or T_1 is in $\mathcal{X}^{dual}, \mathcal{R}^{dual}, \mathcal{S}^{dual}$ or $\mathcal{Q}^{dual}$ then, the interpolated $T_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ is in the same class.

8. Factorization. Let $\mathcal{I}$ be an operator ideal; space $(\mathcal{I})$ is the class of all Banach spaces such that $1_X \in \mathcal{I}$. An operator ideal $\mathcal{I}$ has the factorization property if for every operator $T \in \mathcal{I}(E,F)$ there exists $X \in$ Space $(\mathcal{I})$ and operators $U \in \mathcal{L}(E,X), V \in \mathcal{L}(X,F)$ such that $T = VU$.

Several important classes of operators have the factorization property, e.g., weakly compact, Rosenthal, Banach-Saks and Decomposing. The factorization property for $\mathcal{W}$ was proved by Davis et al. in [DFJP]; for $\mathcal{R}$ and $\mathcal{S}$ by Beauzamy in [B]; for $\mathcal{Q}$ by Heinrich in [H].

Other, also important classes, have not the factorization property; for example Radon-Nikodym and unconditionally summing. This was proved by Ghoussoub and Johnson in [GJ].

Lemma 8.1. Every surjective operator ideal with the strong interpolation property for the real method (depending on $0 < \theta < 1$ and $1 < p < \infty$) has the factorization property.

Proof. Let $T \in \mathcal{L}(E,F)$ and denote by G the space $Im(T\ )$ with the norm obtained from the Minkowski functional of the convex, symmetric and absorbent set $T(B_E)$. This G is a Banach space isometrically isomorphic to the quotient space $E/Ker(T)$ and is imbedded into F. By the surjectivity of the ideal this imbedding $i : G \to F$ belongs to $\mathcal{I}$. Since $\mathcal{I}$ has the strong interpolation property the spaces $(G,F)_{\theta,p}$ with $0 < \theta < 1$ and $1 < p < \infty$ belong to space $(\mathcal{I})$ and T factors through $(G,F)_{\theta,p}$. Indeed, let $\bar{T}$ be the operator $T : E \to G$; if u is the imbedding of G in $(G,F)_{\theta,p}$ and j that of $(G,F)_{\theta,p}$ in F one has that T factors by $u\bar{T} : E \to (G,F)_{\theta,p}$ and $j : (G,F)_{\theta,p} \to F$.

This Lemma together with theorem 7.2 prove the following theorem

Theorem 8.2. The operator ideals $\mathcal{X}^{dual}, \mathcal{R}^{dual}, \mathcal{S}^{dual}$ and $\mathcal{Q}^{dual}$ possess the factorization property.

9. Other classes of operators. Let $\mathcal{A}$ and $\mathcal{B}$ be two operator ideals. The product $\mathcal{B} \circ \mathcal{A}$ is a new operator ideal defined as follows: $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{B} \circ \mathcal{A}$ if there exists a Banach space G and operators $U \in \mathcal{A}(E,G), V \in \mathcal{B}(G,F)$ such that $T = VU$.

S. Heinrich proves in [H], Thm. 1.1 that if $\mathcal{A}$ and $\mathcal{B}$ are closed then $\mathcal{B} \circ \mathcal{A}$ is closed also.

Is clear that if $\mathcal{A}$ is surjective $\mathcal{B} \circ \mathcal{A}$ is surjective and if $\mathcal{B}$ is injective $\mathcal{B} \circ \mathcal{A}$ is injective also. Always is true that $\mathcal{B} \circ \mathcal{A} \subset \mathcal{B} \cap \mathcal{A}$ but the converse inclusion is not valid in general. Nevertheless Heinrich shows in [H], Thm. 1.3 that if $\mathcal{A}$ is injective and $\mathcal{B}$ is surjective then $\mathcal{B} \circ \mathcal{A} = \mathcal{B} \cap \mathcal{A}$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are closed, injective and surjective operator ideals with the strong interpolation property for the real method ($0 < \theta < 1$ and $1 < p < \infty$). Since $\mathcal{B} \circ \mathcal{A}$ is the intersection $\mathcal{B} \cap \mathcal{A}$ this ideal also possess the strong interpolation property.

On the other hand the strong interpolation property es preserved by dualizing (Lemma 7.2).

The classes $\mathcal{X}$,$\mathcal{R}$,$\mathcal{W}$,$\mathcal{S}$ and $\mathcal{Q}$ are closed, injective, surjective and possess the strong interpolation property. Therefore, taking products and dualizing one obtain many new classes of injective, surjective and closed operator ideals with the strong interpolation property for the real method ($0 < \theta < 1$ and $1 < p < \infty$) and, by that, with the factorization property.

Take non trivial chains, as for example $\mathcal{W} \circ \mathcal{X}$. It possesses the strong interpolation property and the factorization property. By that is not only an idempotent operator ideal, see [H], Corollary 1.9.

Perhaps is interesting the following easy proof of the fact that $\mathcal{W} \circ \mathcal{X} = (\mathcal{W} \circ \mathcal{X})^{dual}$.

Proposition 9.1. $\mathcal{W} \circ \mathcal{X}$ is completely symmetric.

Proof. $\mathcal{W} \circ \mathcal{X}$ and $(\mathcal{W} \circ \mathcal{X})^{dual}$, both, possess the factorization property. Since a Banach space is reflexive and separable if and only if its dual is reflexive and separable one concludes.

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