INTERPOLATING SOME CLASSES OF OPERATORS BETWEEN FAMILIES

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In memory of my beloved Uncle, Diógenes,

Abstract

Many ideals of operators (single ideals and chains) possess the strong property of interpolation for the J and K methods of Lions-Peetre, Sparr, Fernández and Cobos-Peetre. That is, let $\mathfrak{I}$ be one of the ideals considered here, let $\overline{A}$ and $\overline{B}$ be interpolation tuples and $T:\overline{A}\to\overline{B}$ a bounded linear operator, then, the interpolated operator $T_{J,K}:J(\overline{A})\to K(\overline{B})$ belongs to $\mathfrak{I}$ if and only if the induced operator $T_{\mathcal{JS}}$ from the intersection $\mathcal{J}(\overline{A})$ into the sum $\mathcal{S}(\overline{B})$ belongs to $\mathfrak{I}$.

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1. Introduction. Let $\mathfrak{I}$ be an operator ideal, let $\overline{A}$ and $\overline{B}$ be finite interpolation tuples, let $J(\overline{A})$ and $K(\overline{B})$ be the spaces obtained from $\overline{A}$ and $\overline{B}$ by the J and K methods of interpolation. The ideal $\mathfrak{I}$ possesses the *Interpolation Property* for a method $\mathcal{F}$ of interpolation (see [4], chap. 2), if the interpolated operator $T_{\mathcal{F}}:\mathcal{F}(\overline{A})\to\mathcal{F}(\overline{B})$ belongs to $\mathfrak{I}$ when all (or some) of the extreme operators $T_i:A_i\to B_i$ $(i=0,1,...,n)$, belong to $\mathfrak{I}$. The ideal possesses the *Strong Property of Interpolation*, SPI in short, respect to the J and K methods for families if the interpolated operator $T_{J,K}:J(\overline{A})\to K(\overline{B})$ belongs to $\mathfrak{I}$ when the induced $T_{\mathcal{JS}}:\mathcal{J}(\overline{A})\to\mathcal{S}(\overline{B})$ from the intersection into the sum spaces is in $\mathfrak{I}$.

The ideals of separable, Rosenthal, weakly compact, Banach-Saks, alternate signs Banach-Saks, decomposing, their dual ideals and chains of them, possess the strong property of interpolation respect to the J and K methods, for finite families, of Lions-Peetre, Sparr, Fernández and Cobos-Peetre.

The paper is firstly concerned with the work of M. J. Carro, ([5], [6], [7], [8]) and that of S. Heinrich, [14]. The roots (of the paper) go back to the famous and celebrated paper of Davis, Figiel, Johnson and Pełczyński, [11].

The main result is achieved in §5, theorem 5.4, after a necessary preamble. This result applys better when J and K methods are equivalent, as in the case of Lions-Peetre method for pairs and as in the many cases of Fernández and Sparr methods when this happens (see [13] and [22]).

Notation is standard; in any case, unexplained symbols, terms, concepts, etc., will be found in [4], [5], [14], [20] or [21].

2. Preliminaries. $\mathfrak{L}(E,F)$ is the space of all bounded linear operators. An operator ideal $\mathfrak{I}$ is a class of bounded linear operators, such that the components $\mathfrak{I}(E,F) = \mathfrak{I} \bigcap \mathfrak{L}(E,F)$ satisfy the following conditions:(i) $\mathfrak{I}(E,F)$ is a linear subspace of $\mathfrak{L}(E,F)$, (ii) $\mathfrak{I}(E,F)$ contains the finite rank operators and (iii) if $R \in \mathfrak{L}(X,E), S \in \mathfrak{I}(E,F)$ and $T \in \mathfrak{L}(F,Y)$ then, $TSR \in \mathfrak{I}(X,Y)$ (see [14] and [20]).

The operator ideal is injective if for every isomorphic embedding $J \in \mathfrak{L}(F,Y)$ one has that $T \in \mathfrak{L}(E,F)$ and $JT \in \mathfrak{I}(E,Y)$ implies $T \in \mathfrak{I}(E,F)$; it is surjective if for every surjection $Q \in \mathfrak{L}(X,E)$ one has that $T \in \mathfrak{L}(E,F)$ and $TQ \in \mathfrak{I}(X,F)$ implies $T \in \mathfrak{I}(E,F)$. The ideal is closed if the components $\mathfrak{I}(E,F)$ are closed subspaces of $\mathfrak{L}(E,F)$ (see [14] and [20], chap. 4).

Every operator ideal $\mathfrak{I}$ defines a class of Banach spaces, *Space*$(\mathfrak{I})$, in the following way: $E \in$ *Space*$(\mathfrak{I})$ if and only if $1_E \in \mathfrak{I}(E,E)$.

Let S be a countable set and $(X_\alpha)_{\alpha\in S}$ a family of Banach spaces; denote by $(\sum_{\alpha\in S} X_\alpha)_p$, with $1 \leq p < \infty$, the space of all the maps $(x_\alpha)_{\alpha\in S}$, such that $x_\alpha \in X_\alpha$ and $\|(x_\alpha)_{\alpha\in S}\| = (\sum_{\alpha\in S} \|x_\alpha\|^p_{X_\alpha})^{1/p} < \infty$.

Denote by J_i the natural embedding of X_i into $(\sum_{\alpha\in S} X_\alpha)_p$ and by Q_j the projection of $(\sum_{\alpha\in S} X_\alpha)_p$ onto X_j.

Definition 2.1. The ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition for $1 \leq p < \infty$ (see [14]), if for any two families $(E_\alpha)_{\alpha\in S}$ and $(F_\alpha)_{\alpha\in S}$ of Banach spaces the following holds: if $T \in \mathfrak{L}((\sum_{\alpha\in S} E_\alpha)_p, (\sum_{\alpha\in S} F_\alpha)_p)$ and $Q_j T J_i \in \mathfrak{I}(E_i, F_j)$ for every $i,j \in S$, then, $T \in \mathfrak{I}((\sum_{\alpha\in S} E_\alpha)_p, (\sum_{\alpha\in S} F_\alpha)_p)$.

3. Real Methods of Interpolation for finite Families. Three real methods of interpolation for finite families generalize the Lions-Peetre method for pairs. They are, in chronological order: the Sparr method (see [22]), the Fernández method for 2^d spaces (see [13]) and the Cobos-Peetre method associated with the vertices of a convex polygon in $\mathbb{R}^2$ (see [9]). In all of them, both, the K and J functionals are defined introducing a positive weight factor $\overline{\omega}$ (tuple of positive, > 0, real numbers) in the norms of the sum and intersection spaces, being $\overline{\omega}$

chosen in a different way for each one.

3.1. M. Carro in [5] and [6], proposed a method of real interpolation for infinite families that, beside the fact of being the natural setting to compare with the complex method for families, proposed by the St. Louis group in [10], provides a unified approach to those methods for finite families. (For a comparison with other real methods for infinite families, see [8]).

In order to briefly describe her method, let D denote the unit disk, $D = \{z \in \mathbb{C} : |z| < 1\}$ and Γ its boundary. The family $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathcal{A}, \mathcal{U}\}$ is a complex *interpolation family* (i.f.) on Γ with $\mathcal{U}$ as the containing space and $\mathcal{A}$ as the log-intersection space if:

(a) for each γ, the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathcal{U}$; $\|\cdot\|_\gamma$ is the norm on $A(\gamma)$ and $\|\cdot\|_{\mathcal{U}}$ that on $\mathcal{U}$,

(b) for every $a \in \bigcap_{\gamma\in\Gamma} A(\gamma)$ the application $\gamma \to \|a\|_\gamma$ is a measurable function on Γ,

(c) If $\mathcal{A}$ is the *log-intersection* linear space:

$$\mathcal{A} = \{a \in A(\gamma) \text{ for a.e. } \gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma d\gamma < \infty\}$$

with $\log^+ = \max(\log, 0)$, then, there exists a measurable function P on Γ such that

$$\int_\Gamma \log^+ P(\gamma) d\gamma < \infty \text{ and } \|a\|_{\mathcal{U}} \le P(\gamma)\|a\|_\gamma \text{ for a.e. } \gamma, (a \in \mathcal{A}).$$

Let $\mathcal{L}$ be the multiplicative group, defined by:

$$\mathcal{L} = \{\alpha : \Gamma \to \mathbb{R}^+; \alpha \text{ is measurable with } \log\alpha \in L^1(\Gamma)\}$$

and $\mathcal{G}$ be the space of all $\mathcal{A}$-valued, simple and measurable functions on Γ. The space $\overline{\mathcal{G}}$ is that of all Bochner integrable (in $\mathcal{U}$) functions $a(\cdot)$ such that $a(\gamma) \in A(\gamma)$ for a.e. $\gamma \in \Gamma$ and that $a(\cdot)$ can be a.e. approximated in the $A(\gamma)$-norm by a sequence of functions from $\mathcal{G}$.

For $\alpha \in \mathcal{L}$ and $a \in \mathcal{U}$, with $a = \int_\Gamma a(\gamma)d\gamma$, define the K-functional with respect to the i.f. $\overline{A}$ by:

$$K(\alpha, a) = \inf\{\int_\Gamma \alpha(\gamma)\|a(\gamma)\|_\gamma d\gamma\},$$

where the infimum is taken over all representations $a = \int_\Gamma a(\gamma)d\gamma$ (convergence in $\mathcal{U}$), with $a(\cdot) \in \overline{\mathcal{G}}$. And, for $a \in \mathcal{A}$ define the J-functional by

$$J(\alpha, a) = \text{ess sup}_{\gamma\in\Gamma}(\alpha(\gamma)\|a\|_\gamma).$$

For $\alpha \in \mathcal{L}$ and $z \in D$, define

$$\alpha(z) = \exp(\int_\Gamma \log \alpha(\gamma) P_z(\gamma) d\gamma),$$

where P_z is the Poisson kernel at $z \in D$, see [5].

Let $\overline{A}$ be an i.f., $S \subset \mathcal{L}$ a multiplicative subgroup and $1 \leq p \leq \infty$. Following Carro's notation, (see, nevertheless, [8], Remark 1.1), define the K-space, $[A]^S_{z_0,p}$, as that of all $a \in \mathcal{U}$ for which

$$\left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)_{\alpha \in S} \in \ell^p(S),$$

endowed with the norm

$$\|a\|_{[A]^S_{z_0,p}} = \left(\sum_{\alpha \in S} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{\frac{1}{p}};$$

as always for $p = \infty$.

The J-space, $(A)^S_{z_0,p}$, is defined as that of all $a \in \mathcal{U}$ for which there exists a map, $(u(\alpha))_{\alpha \in S}$, from S into $\mathcal{A}$, so that $a = \sum_{\alpha \in S} u(\alpha)$ (convergence in the $\mathcal{U}$ norm) and

$$(*) \qquad \left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)} \right)_{\alpha \in S} \in \ell^p(S),$$

endowed with the norm

$$\|a\|_{(A)^S_{z_0,p}} = \inf \left(\sum_{\alpha \in S} \left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)} \right)^p \right)^{\frac{1}{p}},$$

where the infimum extends over all representations of a.

In order to have that they are Banach intermediate spaces and the embedding of $(A)^S_{z_0,p}$ into $[A]^S_{z_0,p}$, some natural conditions are necessary on S. These conditions are (see [5] and [6], §2, page 56):

(i)For every $\alpha \in S$ there exists a constant C_α such that $P(\gamma) \leq C_\alpha \alpha(\gamma),$ a.e. γ (see the definition of i.f.).
(ii)For every $z_0 \in D$, there exists a compact $K \subset D$ such that

$$\sum_{\alpha \in S} \frac{\inf_{z \in K} \alpha(z)}{\alpha(z_0)} < \infty$$

.
(iii) S is a multiplicative group.

Under these conditions, $(*)$ implies the absolute convergence of $\sum_{\alpha \in S} u(\alpha)$ in $\mathcal{U}$ (see [5], §4, Proposition 4.3 and Remark 4.4).

Let $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathcal{A}, \mathcal{U}\}$ and $\overline{B} = \{B(\gamma) : \gamma \in \Gamma; \mathcal{B}, \mathcal{V}\}$ be two i.f.; let $T : \overline{A} \to \overline{B}$ be an interpolation operator, i.e., $T : \mathcal{U} \to \mathcal{V}$ is a bounded linear operator and $T_\gamma : A(\gamma) \to B(\gamma)$ bounded for each γ with $\|T_\gamma\|_{A(\gamma)\to B(\gamma)} \leq M(\gamma) \in \mathcal{L}$. If $\|M\|_\infty < \infty$, then, for every $S \subset \mathcal{L}$:

$$T^S_{z_0,p} : [A]^S_{z_0,p} \to [B]^S_{z_0,p} \text{ and } T^S_{z_0,p} : (A)^S_{z_0,p} \to (B)^S_{z_0,p}$$

are bounded with norms $\leq \|M\|_\infty$.

In view that $(B)^S_{z_0,p}$ is embedded into $[B]^S_{z_0,p}$, the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ is also bounded.

3.2. Now, consider finite families of spaces A_i, continuously embedded into the same Hausdorff topological vector space $\mathcal{H}$. As in [2], denote by $\mathcal{J}(\overline{A})$ the intersection $\bigcap A_i$ and by $\mathcal{S}(\overline{A})$ the sum $A_0 + A_1 + \ldots + A_n$, with the norms

$$\|a\|_{\mathcal{J}(\overline{A})} = \max\{\|a\|_{A_0}, \|a\|_{A_1}, \ldots, \|a\|_{A_n}\}$$

and

$$\|a\|_{\mathcal{S}(\overline{A})} = inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} + \ldots + \|a_n\|_{A_n}\},$$

where the infimum extends over all representations of $a = a_0 + a_1 + \ldots + a_n$. Suppose that $\mathcal{J}(\overline{A})$ is dense in every A_i (see [5], §3, Proposition 3.7; [6], §2, Proposition 2.6), and do $\mathcal{A} = \mathcal{J}(\overline{A}), \mathcal{U} = \mathcal{S}(\overline{A})$:

(i) Let $\overline{A} = (A_0, A_1)$; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, i=0,1, with $\{\Gamma_0, \Gamma_1\}$ a partition of Γ. Do

$$S = \{\alpha_m = 1_{\Gamma_0} + 2^m 1_{\Gamma_1}; m \in \mathbb{Z}\},$$

to get that $[A]^S_{z_0,p} = (A_0, A_1)_{|\Gamma_1|_{z_0},p} = K_{\theta,p}(\overline{A})$: the K-space of Lions-Peetre with $\theta = |\Gamma_1|_{z_0}$, where $|E|_z$ is the harmonic measure of $E \subset \Gamma$ at $z \in D$, (see [15]).

(ii) Let $\overline{A} = (A_0, A_1 \ldots, A_n)$; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, $i = 0, 1, \ldots, n$ and $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ a partition of Γ. Do

$$S = \{\alpha_{\overline{m}} = 1_{\Gamma_0} + \sum_{i=1,n} 2^{m_i} 1_{\Gamma_i}; \overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n\},$$

to get that $[A]^S_{z_0,p} = (A_0, A_1 \ldots, A_n)_{(|\Gamma_i|_{z_0}, i=1,\ldots,n),p;K}$: the Sparr K-space, see [22].

(iii) Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a family of 2^2 spaces; take $A(\gamma) = A_i$ with $\gamma \in \Gamma_i$, i=0,1,2,3 and $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$ a partition of Γ. Do

$$S = \{\alpha_{\overline{m}} = 1_{\Gamma_0} + 2^k 1_{\Gamma_1} + 2^l 1_{\Gamma_2} + 2^k 2^l 1_{\Gamma_3}; \overline{m} = (k,l) \in \mathbb{Z}^2\},$$

to obtain that $[A]^S_{z_0,p} = (A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2),p;K}$, where $\theta_1 = |\Gamma_1 \bigcup \Gamma_3|_{z_0}$, $\theta_2 = |\Gamma_2 \bigcup \Gamma_3|_{z_0}$: Fernández K-space, which can be generalized to families of 2^d spaces, see [13].

(iv) Let $\overline{A} = (A_1, \ldots, A_n)$; do

$$S = \{\alpha_{(k,l)} = \sum_{i=1,n} 2^{kx_i + ly_i} 1_{\Gamma_i}; (k,l) \in \mathbb{Z}^2\},$$

where (x_i, y_i) are the vertices of a convex polygon Π in the affine plane $\mathbb{R}^2$, to obtain, for an interior point (α, β) of Π that $[A]^S_{z_0,p} = \overline{A}_{(\alpha,\beta),p;K}$, with $(\alpha,\beta) = \sum_{i=1,n} |\Gamma_i|_{z_0}(x_i, y_i)$: Cobos-Peetre K-space, see [9].

With the same S in each case, apply the J-method just described to obtain the J-spaces, $(A)^S_{z_0,p}$, of Lions-Peetre, Sparr, Fernández and Cobos-Peetre, respectively. The density of $\mathcal{J}(\overline{A})$ in each A_i is not necessary for the J-method.

Despite the fact that J-space always embedds into K-space, J and K-methods are not equivalent. Nevertheless, they are equivalent in the case of Lions-Peetre method for pairs, see [4], and in many and very important cases of Fernández and Sparr methods, see [13] and [22].

4. Some Classes of Operators.

4.1. An operator $T \in \mathfrak{L}(E, F)$ is compact if the image $T(B_E)$ of the unit ball of E is relatively compact in the norm topology of F. The operator is weakly compact if $T(B_E)$ is a relatively weakly compact set, or, using the Eberlein-Smulian Theorem, if and only if every sequence (Tx_n) with $x_n \in B_E$ admits a weakly convergent subsequence. The operator T is separable if $T(E)$ is a separable subspace of F or, equivalently, if $T(B_E)$ is a separable subset of F.

$T \in \mathfrak{L}(E, F)$ is a Rosenthal operator if for each $s \in \mathfrak{L}(\ell_1, E)$ the composition Ts is not an isomorphic embedding; T is unconditionally summing if for each $s \in \mathfrak{L}(c_0, E)$ the composition Ts is not an isomorphic embedding. In other words, T does not 'transport' copies of ℓ_1 or c_0, respectively. Using Rosenthal and Bessaga-Pełczyński theorems, it is easy to obtain the following characterization of these operators: T is Rosenthal if and only if every bounded sequence (x_n) of E possesses a subsequence (x_{n_k}) such that (Tx_{n_k}) is weak Cauchy, that is, if and only if $T(B_E)$ is weakly pre-compact. T is unconditionally summing if and only if for every sequence (x_n) of E which is unconditionally summable (i.e., for every sequence (x_n) such that $\sum_{n=1,\infty} \mid f(x_n) \mid < \infty$ for all $f \in E^*$) the sequence (Tx_n) is unconditionally summable in the norm topology of F, see [17].

$T \in \mathfrak{L}(E,F)$ has the Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n), such that (Tx'_n) is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum_{k=1,n} Tx'_k$, converges in F. The operator T has the alternate-signs Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n), such that $((-1)^n Tx'_n)$ is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum_{k=1,n} (-1)^k Tx'_k$ converges in F. The operator T has the Banach-Saks-Rosenthal property if any weakly null sequence (x_n) of E possesses a subsequence (x'_n), such that the sequence of the averages $n^{-1} \sum_{k=1,n} Tx'_k$ converges in F. See [1] for a thorough study of these operators, and also see [16].

Let (Ω, μ) be a probability space. An operator $X \in \mathfrak{L}(L_1(\Omega,\mu), E)$ is right-decomposable if there exists a μ-measurable and E-valued kernel $x(\omega)$, with $\omega \in \Omega$, such that for all $f \in L_1(\Omega,\mu)$:

$$X(f) = \int_\Omega f(\omega)x(\omega)d\mu.$$

$T \in \mathfrak{L}(E,F)$ is a Radon-Nikodym operator if TX is right-decomposable for every $X \in (L_1(\Omega,\mu), E)$.

An operator $Z \in \mathfrak{L}(F, (L_\infty(\Omega,\mu))$ is left-decomposable if there exists a μ-measurable and F^*-valued kernel $z(\omega)$, such that for all $y \in F$:

$$Zy(\omega) = \langle y, z(\omega) \rangle.$$

$T \in \mathfrak{L}(E,F)$ is a decomposing operator if ZT is left decomposable for every $Z \in \mathfrak{L}(F, (L_\infty(\Omega,\mu))$; see [20], Chap. 24.

The following characterization of decomposing operators is well known, see [20]: T is decomposing if and only if its dual, T^*, is a Radon-Nikodym operator.

All the classes of operators defined above are ideals of operators in the sense of Pietsch. As in [20], capital gothic letters will denote each one of them. So: $\mathfrak{K}$ will be the ideal of compact operators, $\mathfrak{W}$ that of weakly compact, $\mathfrak{X}$ separable operators, $\mathfrak{R}$ Rosenthal operators, $\mathfrak{U}$ unconditionally summing, $\mathfrak{Y}$ Radon-Nikodym and $\mathfrak{Q}$ that of decomposing operators; see [20] for a detailed study of these ideals.

Place $\mathcal{BS}$ for Banach-Saks operator ideal, $\mathcal{ABS}$ for alternate-signs Banach-Saks and $\mathcal{BSR}$ for Banach-Saks-Rosenthal operator ideal. These last ideals are not treated by Pietsch; see [1] for their study.

All these ideals are closed and injective. Compact, weakly compact, separable, Rosenthal, Banach-Saks, alternate-signs Banach-Saks and decomposing are also surjective operator ideals. Neither $\mathfrak{U}$ nor $\mathfrak{Y}$ nor $\mathcal{BSR}$ are surjective.

A bounded subset A of the space X is called limited if $\lim_{n\to\infty} \sup_{x \in A} |x^*_n(x)| = 0$ for every *weak**-null sequence (x^*_n) in X^*, i.e., $\lim_{n\to\infty} x^*_n(x) = 0$ uniformly on A. The operator $T \in \mathfrak{L}(E,F)$ is limited if $T(B_E)$ is a limited subset of F. Clearly,

T is limited if and only if $T^* : F^* \to E^*$ takes *weak**-null sequences to norm null sequences. See [3] and [7] for a detailed study of these operators.

4.2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two operator ideals. The product $\mathfrak{D} \circ \mathfrak{C}$ is a new operator ideal defined as follows: $T \in \mathfrak{L}(E,F)$ belongs to $\mathfrak{D} \circ \mathfrak{C}$ if there exists a Banach space G and operators $U \in \mathfrak{C}(E,G), V \in \mathfrak{D}(G,F)$, such that $T = VU$.

Heinrich, in [14], Thm. 1.1, proves that if $\mathfrak{C}$ and $\mathfrak{D}$ are closed then $\mathfrak{D} \circ \mathfrak{C}$ is also closed. It is always true that $\mathfrak{D} \circ \mathfrak{C} \subset \mathfrak{D} \cap \mathfrak{C}$, but the converse inclusion is not valid in general. Nevertheless, see [14], Thm. 1.3, if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective then $\mathfrak{D} \circ \mathfrak{C} = \mathfrak{D} \cap \mathfrak{C}$.

This product is, certainly, associative, non commutative in general, although, if $\mathfrak{C}$ and $\mathfrak{D}$ are closed, injective and surjective operator ideals, the product commutes. The identity element is, of course, $\mathfrak{L}$.

Also, is clear that if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective, both satisfying the $\sum_p$-condition, $1 \leq p < \infty$, then, $\mathfrak{D} \circ \mathfrak{C} = \mathfrak{D} \cap \mathfrak{C}$ satisfies the $\sum_p$-condition.

Let $\mathfrak{I}$ be an operator ideal. The operator $T \in \mathfrak{L}(E,F)$ belongs to the dual ideal $\mathfrak{I}^{dual}$ if the adjoint operator T^* belongs to $\mathfrak{I}(F^*, E^*)$. For example, the fact that T is decomposing if and only if its dual, T^*, is a Radon-Nikodym operator, means that $\mathfrak{Q} = \mathfrak{Y}^{dual}$.

If $\mathfrak{I}$ is injective, $\mathfrak{I}^{dual}$ is surjective and if $\mathfrak{I}$ is surjective, $\mathfrak{I}^{dual}$ is injective (see [20], chap. 4). If $\mathfrak{I}$ is closed, so is $\mathfrak{I}^{dual}$. If $\mathfrak{I}$ satisfies the $\sum_p$-condition, for all p with $1 < p < \infty$, then $\mathfrak{I}^{dual}$ satisfies $\sum_p$-condition for all p with $1 < p < \infty$.

Some times happens that, depending on the relationship between the ideals, the product reduces, as for example, the product $\mathfrak{W} \circ \mathcal{BS} \circ \mathfrak{R}$ which is equal to $\mathfrak{R}$, or, as in the case of weakly compact operators, for which $\mathfrak{W}^{dual} = \mathfrak{W}$, etc., (see [20]).

The product of several operator ideals will be called a *chain*. For example, $\mathfrak{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathfrak{I} = (\mathfrak{Q} \circ \mathcal{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, all them being, certainly, injective, surjective and closed operator ideals, are chains.

Let E and F be Banach spaces. An operator $T \in chain(E,F)$ is a *mixed operator*. For example, $T \in \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}(E,F)$, which means that $T : E \to F$ is, **at the same time**, a separable, Rosenthal and dual Radon-Nikodym operator, is a mixed operator of type $\mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$.

4.3. The relationships $\mathcal{BS} \subset \mathcal{ABS} \subset \mathcal{WBS}$, strict inclusions, are well known. The following theorem gives relationships much more precise that will be used in the next sections:

Theorem 4.4. *i)*$\mathcal{BS} = \mathfrak{W} \circ \mathcal{BSR}$, *ii)*$\mathcal{BS} = \mathfrak{W} \circ \mathcal{ABS}$ and *iii)*$\mathcal{ABS} = \mathfrak{R} \circ \mathcal{BSR}$ (the products must be in this order because $\mathcal{BSR}$ is not surjective).

Proof. *i)* for $\mathfrak{W} \circ \mathcal{BSR} \subset \mathcal{BS}$ use the Eberlein-Smulian characterization of weakly compact operators; for $\mathcal{BS} \subset \mathfrak{W} \circ \mathcal{BSR}$, use that $\mathcal{BS} \subset \mathfrak{W}$, (see [2], Lemma 1, page 49). Item *ii)* follows at once from *i)*. Item *iii)* is exactly the reformulation for operator ideals of the well known fact that for Banach spaces without copies of ℓ_1, $\mathcal{ABS}$ and $\mathcal{BSR}$ properties, are equivalent, (see [1], Section II, Prop. 3); the proof of *iii)* is the same given there, with the obvious modifications./

Now, the objective will be interpolate mixed operators.

5. Interpolation of Operators. Interpolation theory is concerned with the following: let $\overline{A}$ and $\overline{B}$ be interpolation tuples and $T : \overline{A} \to \overline{B}$ an interpolation operator, if the extreme operators $T_i : A_i \to B_i$ $(i = 0, 1, ..., n)$, or some of them, belong to a class $\mathfrak{I}$ of operators, what can be expected from the interpolated $T^S_{z_0,p}$? (see [21]).

If $T : \overline{A} \to \overline{B}$ is an interpolation operator, denote by $T_{\mathcal{J}\mathcal{S}}$ the induced operator from $\mathcal{J}(\overline{A})$ into $\mathcal{S}(\overline{B})$.

Definition 5.1. An operator ideal $\mathfrak{I}$, possesses the *Strong Property of Interpolation* (SPI, in short), respect to the Carro's Method, i.e., the real method for families depending on the parameters $z_0 \in D, 1 \leq p < \infty$ and $S \subset \mathcal{L}$, if the following holds: the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ belongs to $\mathfrak{I}$ if and only if $T_{\mathcal{J}\mathcal{S}} \in \mathfrak{I}$.

Theorem 5.2. Any (closed), injective and surjective operator ideal, $\mathfrak{I}$, which satisfies the $\sum_p$-condition, possess SPI, see [21].

Proof. Let $\overline{A}$ and $\overline{B}$ be an i.f. of finite tuples. In order to avoid a complicated notation, write A for the intersection $\mathcal{J}(\overline{A})$ and B for the sum $\mathcal{S}(\overline{B})$. Define on A and B the following equivalent norms (equivalent to the norms of intersection and sum spaces, respectively):

$$\|x\|_\alpha = \frac{J(\alpha, x)}{\alpha(z_0)} \quad \text{for } x \in A \text{ and } \alpha \in S,$$

$$\|y\|_\alpha = \frac{K(\alpha, y)}{\alpha(z_0)} \quad \text{for } y \in B \text{ and } \alpha \in S,$$

where S is the corresponding multiplicative subgroup of $\mathcal{L}$ for each method described in §**3.2.**

Denote by A_α the space $(A, \| \ \|_\alpha)$ and by B_α the space $(B, \| \ \|_\alpha)$. For each $(x_\alpha)_{\alpha\in S} \in (\sum_{\alpha\in S} A_\alpha)_p$, the sum $\sum_{\alpha\in S} x_\alpha$ converges (absolutely) in $\mathcal{S}(\overline{A})$. Then, there is a surjection Q from $(\sum_{\alpha\in S} A_\alpha)_p$ onto the J-space $(A)^S_{z_0,p}$:

$$Q(x_\alpha)_{\alpha\in S} = \sum_{\alpha\in S} x_\alpha \quad (\text{convergence in } \mathcal{S}(\overline{A}))$$

and an isomorphic embedding J from the K-space $[B]^S_{z_0,p}$ into $(\sum_{\alpha\in S} B_\alpha)_p$ defined by $J(y) = (y_\alpha)_{\alpha\in S}$ such that $y_\alpha = y$ for all α.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator and assume that $T_{\mathcal{J}\mathcal{S}} \in \mathfrak{I}$, denote by J_i the embedding of A_i into $(\sum_{\alpha\in S} A_\alpha)_p$ and by Q_j the projection of

$(\sum_{\alpha\in S} B_\alpha)_p$ onto B_j. The operator Q_jJTQJ_i is just $T_{\mathcal{JS}}$. It is, then, an operator of the class $\mathfrak{I}$ and, since $\mathfrak{I}$ satisfies the $\sum_p$-condition, the operator JTQ belongs to $\mathfrak{I}((\sum_{\alpha\in S} A_\alpha)_p, (\sum_{\alpha\in S} B_\alpha)_p)$. Now, injectivity and surjectivity of $\mathfrak{I}$ imply that $T^S_{z_0,p} \in \mathfrak{I}((A)^S_{z_0,p}, [B]^S_{z_0,p})$. Converse is clear./

Theorem 5.3. The single ideals $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BSR}, \mathcal{BS}, \mathcal{ABS}$ and $\mathfrak{Q}$ satisfy the $\sum_p$-condition for $1 < p < \infty$.

Proof. Let $\mathfrak{I}$ be one of these ideals. First, prove that if $(X_n)_{n\in\mathbb{N}}$ is a family of Banach spaces such that each X_n possesses the space property defined by $\mathfrak{I}$ then $(\sum_{n\in\mathbb{N}} X_n)_p, 1 < p < \infty$, also possesses this property. For $\mathcal{BS}$ and $\mathcal{BSR}$, this is a well known Theorem of J. R. Partington, see [19].

Let $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ be two families of Banach spaces, assume that $T \in \mathfrak{L}((\sum_{n\in\mathbb{N}} E_n)_p, (\sum_{n\in\mathbb{N}} F_n)_p), 1 < p < \infty$ and that $Q_jTJ_i \in \mathfrak{I}(E_i, F_j)$ for every i, j; try to prove that $T \in \mathfrak{I}((\sum_{n\in\mathbb{N}} E_n)_p, (\sum_{n\in\mathbb{N}} F_n)_p)$ in two steps, for finite families first and, after, the general case:

(i) Easy for $\mathfrak{X}$, including the case $p = 1$.

(ii) For $\mathfrak{W}$ and $\mathfrak{R}$, use Eberlein-Smulian and Rosenthal Theorems. By a diagonal argument obtain, for all bounded sequence $[(x^k_n)_{n\in\mathbb{N}}]_{k\in\mathbb{N}}$ from $(\sum_{n\in\mathbb{N}} E_n)_p$, a subsequence $[(x^{k_i}_n)_{n\in\mathbb{N}}]_{i\in\mathbb{N}}$ such that $(T[(x^{k_i}_n)_{n\in\mathbb{N}}])_{i\in\mathbb{N}}$ is weakly convergent or weakly Cauchy, respectively, in $(\sum_{n\in\mathbb{N}} F_n)_p$. Recall, for $1 < p < \infty : (\sum_{n\in\mathbb{N}} F_n)^*_p = (\sum_{n\in\mathbb{N}} F^*_n)_{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii) For $\mathcal{BSR}$ (and $\mathcal{BS}$), the proof is that given by Heinrich in [14]. Use the following fact for the verification of the $\sum_p$-condition: the ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition if the following holds for arbitrary Banach spaces E, F and $G_n, (n = 1, 2, \ldots)$: $U \in \mathfrak{L}(E, (\sum_{n\in\mathbb{N}} G_n)_p), V \in \mathfrak{L}((\sum_{n\in\mathbb{N}} G_n)_p), F)$ and $VJ_nQ_nU = VP_nU \in \mathfrak{I}(E, F)$ for all n, implies $VU \in \mathfrak{I}(E, F)$.

Take a weakly null sequence (x_n) from E (respectively, a bounded sequence), use the Erdös-Magidor result on regular methods of summability (see [12]), a diagonal argument, and reason with Heinrich to conclude that there exists a subsequence (x'_n) which is Cesàro convergent, see [14], p. 407.

(iv) For $\mathcal{BS}$ and $\mathcal{ABS}$, use Theorem 4.4., recall that if each ideal satisfy the $\sum_p$-condition, the product also satisfies it and, finally, apply (ii) and (iii).

(v) For $\mathfrak{Q}$, see [14], p. 408.

Excepting the case of $\mathfrak{X}$, the assumption that $1 < p < \infty$ is, clearly, necessary./

It is clear that all the chains proposed as examples in §4.2., satisfy $\sum_p$-condition with $1 < p < \infty$.

Limited operators do not satisfy the $\sum_p$-condition for any p; they are like compact: according to the Josefson-Nissenzweig Theorem (see [3]), if the unit ball of a Banach space is a limited set, the space is finite dimensional.

Now is clear the next theorem. Except for $\mathcal{ABS}$, this result was obtained in [21] for the special case of the Lions-Peetre method for pairs:

Theorem 5.4. The single ideals $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathcal{BS}, \mathcal{ABS}, \mathfrak{Q}$, dual ideals $\mathfrak{X}^{dual}, \mathfrak{R}^{dual}$, $\mathcal{BS}^{dual}, \mathcal{ABS}^{dual}, \mathfrak{Q}^{dual}$ and chains as $\mathfrak{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathfrak{I} = (\mathfrak{X} \circ \mathcal{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, all possess the SPI.
proof. They are injective, surjective and satisfy $\sum_p$-condition for $1 < p < \infty$ (satisfying $\sum_p$-condition they are closed)./

Be clear, the ideal $\mathfrak{R}^{dual}$, for example, possesses SPI means that the interpolated operator, $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$, belongs to $\mathfrak{R}^{dual}$, (that is, its adjoint is a Rosenthal operator), if and only if the induced $T_{\mathcal{JS}}$ belongs to $\mathfrak{R}^{dual}$; the chain $\mathfrak{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, for example, possesses SPI, means that the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ is, **at the same time**, a separable, dual Rosenthal and decomposing operator if and only if the induced $T_{\mathcal{JS}}$ is, **at the same time**, a separable, dual Rosenthal and decomposing operator.

See [7], for a thorough study of SPI for weakly compact operators between infinite families (in general, Thm. 5.1.1), finite families (in particular) and for interpolation of limited operators.

In the case of $\mathcal{ABS}$, Theorem 5.4. generalizes that result obtained by A. Kryczka, in [16], Corollary 4.2..

As was said before, the results of this section apply better in the cases where J and K methods for families are equivalent, see [13] and [22].

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