An Introduction to Realism

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Preliminaries

This manuscript provides an elementary introduction to a new wave theory called Realism, which is different from Quantum Mechanics. Realism is a classical wave theory with states, a deterministic and continuous evolution equation and observables which are continuous functions of state. The evolution is non-linear and is endowed with sensitive dependency on initial conditions. The non-linearity of Realism is intrinsic and originates in the non-linear nature of the space of states. This is *not* the kind of non-linearity that refers to states in a *linear space* evolving according to a *non-linear* expression that involves the state and its derivatives.

Previous study of, and adherence to, Quantum Mechanics creates preconceptions that becloud the natural and simple physical ideas underlying Realism. Readers already compromised with Quantism are invited to set aside their well intentioned acceptance of that theory, at least for the moment, and attempt to assimilate the new perspective on microphysics provided here. They will be able to verify that there is no use for the Uncertainty Principle, Wave-Particle Duality, Probabilistic Interpretation of Waves, Probabilistic Quantum Jumps, Occult Variables and similar constructs of Quantism, while the eigenvalue and eigenfunction calculations are retained and play the expected crucial role.

It is assumed that the reader is familiar with calculus, linear algebra, elementary linear differential equations and their most rudimentary applications in the context of Hamiltonian operators, eigenvalues, eigenfunctions and evolution equations. The exposition is organized as follows.

The three general principles of Realism are are stated and their use is exemplified with a discussion of a 2-level system. The principles are the definitions of space of states, energy observable and evolution equation. Consequences of the postulates, still for the 2-level case, are explained. A rather plausible physical hypothesis, the Photon Hypothesis, is stated in section **14**. It simply says that the radiated energy travels with fixed direction and with constant speed c. Together with the principles of Realism, the Photon Hypothesis provides a partial resolution of the photon structure.

Sensitive dependency on initial conditions, sometimes called *chaos*, appears in three level systems, a case not discussed here but explained in papers available at the URL mentioned below. This is perhaps the mildest form of chaos a continuous system can present but provides the reason why a system in an excited stationary state makes transitions to one of *several* possible stationary states with lower energy. In classical mechanics the throw of a coin is in principle a totally deterministic phenomenon that can be better approached with a simple-minded probabilistic model. To predict the outcome requires, besides the solution of a differential equation, a precise knowledge of the initial state. A similar situation occurs in Realism, with a difference: A single, isolated hydrogen atom is formally a much simpler system than a macroscopic coin and therefore the relevant equation, Schrödinger eigenvalue equation, is easier to solve.

Risking to be tiresome, concision has been sacrificed and we have relied on a redundant and repetitive style with display of many formulas and figures. An alternative presentation of Realism can be found in

`http://euler.ciens.ucv.ve/~dcrespin/Pub`

A discussion of spin dynamics is contained in the paper *Projective Spectral Theorems and Deterministic Atoms.*

Regardless of the generality of the presentation or the arguments given, the Hamiltonian of the hydrogen atom, brilliantly proposed and analyzed by Erwin Schrödinger, but later misinterpreted by Quantists, is both the guiding principle and the final goal of our analysis.

Sociological issues associated with scientific innovations are explained in *The Structure of Scientific Revolutions* by Thomas Kuhn; if his arguments are correct the eventual replacement of Quantism should occur against considerable initial resistance and presumably will only happen at the hands of the younger generation.

General Principles and Two level systems

1.- HAMILTONIANS AND WAVE FUNCTIONS: A system is specified by its Hamiltonian. The Hamiltonian is a symmetric linear operator defined on some collection of wave functions. These wave functions are *real valued* functions of their arguments and have continuous derivatives. Wave functions can be added together and multiplied by real scalars forming a *vector space* E over the real scalars. Additionally, functions can be multiplied together and their inner product is $\langle\psi,\phi\rangle = \int \psi\,\phi$. Recall that $\langle\psi_1+\psi_2,\phi\rangle = \langle\psi_1,\phi\rangle + \langle\psi_2,\phi\rangle$, $\langle\lambda\psi,\phi\rangle = \lambda\langle\psi,\phi\rangle = \langle\psi,\lambda\phi\rangle$, $\langle\psi,\phi\rangle = \langle\phi,\psi\rangle$ and $\langle\psi,\psi\rangle \geq 0$.

The Hamiltonian transforms a given wave function ψ into a wave function $H(\psi)$. Symmetry of H means that $\langle H(\psi),\phi\rangle = \langle\psi,H(\phi)\rangle$ and this implies, for example, $\langle H(\psi),H(\phi)\rangle = \langle H^2(\psi),\phi\rangle = \langle\psi,H^2(\phi)\rangle$. Here $H^2(\psi) = H(H(\psi))$. An *eigenfunction* is a non zero wave function ψ_i such that $H(\psi_i) = -\lambda_i\,\psi_i$ for some (real) scalar $-\lambda_i$ (the minus sign is conventional); this scalar is an *eigenvalue* of H. One then says that the eigenfunction ψ_i and the eigenvalue $-\lambda_i$ *belong* to each other. The Kronecker symbol is $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Beyond these properties, explicit formulas for the eigenfunctions are not required. If the Hamiltonian has a complex valued eigenfunction $\psi_k + \imath\phi_k$ belonging to the eigenvalue $-\lambda_k$ it can be assumed that $\langle\psi_k,\phi_k\rangle = 0$ and then the real and imaginary parts, ψ_k and ϕ_k, are real valued eigenfunctions belonging to the eigenvalue $-\lambda_k$. In this way complex valued eigenfunctions give rise to real valued ones.

Wave functions ψ of interest are linear combinations of eigenfunctions $\psi = a_0\psi_0 + \cdots + a_n\psi_n$; the real number a_i is the *ith-coefficient.* Series, that is, infinite linear combinations $\sum_{i=0}^{\infty} a_i\psi_i$, are usually needed to deal with systems having infinitely many levels.

Let a given system be specified by a Hamiltonian H defined on a collection E of wave functions. The Hamiltonian H' is a subsystem of H if it is defined in a subcollection E' of E and the following two properties hold: a) If wave function ψ' is in E' then $H(\psi')$ is also in E. b) $H(\psi') = H'(\psi')$. This is usually paraphrased saying that H' is the restriction of H to an invariant subspace.

The *number of levels* of H is the number of different eigenvalues. Therefore a system can have 1, 2, 3, $\cdots$, n or infinitely many levels; in the infinite case it will be assumed that eigenvalues form a increasing sequence. Systems with a continuum of eigenvalues can also be studied.

Eigenfunctions belonging to the eigenvalue $-\lambda_i$, together with the zero function, form a linear subspace of E, the eigenspace $E(-\lambda_i)$ of $-\lambda_i$. This means that a linear combination of eigenfunctions belonging to an eigenvalue, and in particular a multiple of an eigenfunction, is again an eigenfunction of the same eigenvalue, or is zero. The dimension of $E(-\lambda_i)$, denoted $m(-\lambda_i)$, is the *multiplicity* or *degenerancy* of $-\lambda_i$. The system is *non-degenerated* or *simple* if all its eigenspaces are 1-dimensional, that is, if for all i one has $m(-\lambda_i) = 1$. Otherwise the system is *degenerated.* Systems discussed in the present paper have 2 levels and are non-degenerated. For a technical discussion that includes the degenerate case see the paper *Stability of Dynamical Systems and Quantum Mechanics* at the URL mentioned above.

For non-degenerated systems a convenient way to define a subsystem H' is to indicate a collection of eigenvalues of H. Subsystems can then be specified by indicating a collection of (mutually orthogonal) eigenfunctions.

A 1-level system has a Hamiltonian with a unique eigenvalue $-\lambda_0$ and a unique eigenfunction ψ_0; all wave functions have the form $\mu\psi_0$ for some scalar μ and H is a 1×1 matrix with single element $-\lambda_0$.

For a 2-level bound system the Hamiltonian has the form

$$H = \begin{bmatrix} -\lambda_0 & 0 \\ 0 & -\lambda_1 \end{bmatrix} \tag{1}$$

with eigenvalues $-\lambda_0 < -\lambda_1 < 0$ and eigenfunctions ψ_0, ψ_1 satisfying $\langle\psi_0, \psi_0\rangle = \langle\psi_1, \psi_1\rangle = 1$ and $\langle\psi_0, \psi_1\rangle = 0$ or, equivalently, $\langle\psi_i, \psi_j\rangle = \delta_{ij}$. In this 2-level case wave functions have the form $\psi = a_0\psi_0 + a_1\psi_1$. As already indicated, complex valued functions and complex coefficients are not needed and will not be used. Wave functions of a 2-level system can be represented as points in a plane with rectangular coordinates a_0 and a_1; see Figure 1.

Examples of eigenvalues are $-\lambda_0 = -13.64$, $-\lambda_1 = -3.41$. Recall that the ground state of the hydrogen atom has energy -13.6 electronvolts and the first excited stationary state has energy $-13.6/2^2 = -3.4$ eV, with other stationary energy values equal to $-13.6/n^2$ eV. Examples of eigenfunctions are eigenfunctions of the hydrogen atom belonging to the eigenvalues.

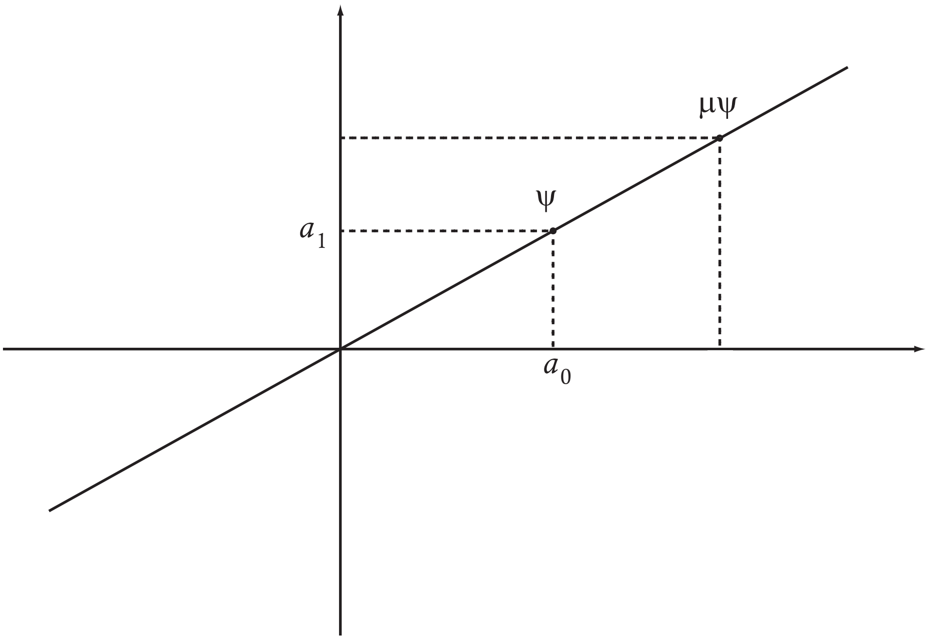

Figure 1. *Each 2-level wave function corresponds to a point in the plane. The line passing through wave functions ψ and $\mu\,\psi$ contains the origin.*

A 2-level system has two 1-level subsystems. These have Hamiltonians given by 1×1 matrices with unique element $-\lambda_0$ or $-\lambda_1$ and the corresponding wave functions are real multiples of ψ_0 or ψ_1. This pair of subsystems correspond with the pair of eigenspaces.

2.- STATES: Careful distinction has to be made between wave functions and states. A *non identically zero* wave function ψ belongs to, and determines, a state $[\psi]$. The following definition of state is the first fundamental principle of Realism. It provides a basic guideline for formulas and calculations.

> DEFINITION OF STATE: Proportional wave functions belong to the same state
>
> $$[\psi] = [\mu\psi] \text{ for all } \mu \neq 0 \tag{2}$$
>
> Conversely, if two wave functions belong to the same state then they are proportional.

Thus, a state is a collection of non-zero, mutually proportional wave functions; will refer to this definition as the *Proportionality Principle.* The factor μ is always a real number. All physically relevant notions defined for states by means of wave functions should be independent of the proportionality factor μ.

Each eigenfunction ψ_i determines a *stationary state* or *eigenstate* $[\psi_i]$. It will be seen that stationary states are in fact motionless: Under the evolution equation to be postulated in section **8** they remain fixed in the space of states.

As indicated in section **1**, wave functions ψ in a 2-level system can be expressed as linear combinations with *real* coefficients $\psi = a_0\psi_0 + a_1\psi_1$, with at least one non-zero coefficient, or, equivalently, $a_0^2 + a_1^2 \neq 0$. Therefore states in a 2-level system have the form $[\psi] = [a_0\psi_0 + a_1\psi_1]$ and the proportionality principle says that

$$[a_0\psi_0 + a_1\psi_1] = [\mu a_0\psi_0 + \mu a_1\psi_1] \text{ for all } \mu \neq 0 \tag{3}$$

For example $[2\psi_0 + 3\psi_1] = [-(2\psi_0 + 3\psi_1)] = [2\sqrt{2}\,\psi_0 + 3\sqrt{2}\,\psi_1] = [(2/3)\psi_0 + \psi_1]$. Here μ has been given the values -1, $\sqrt{2}$ and $1/3$.

Since 2-level wave functions ψ correspond with points in a plane, states $[\psi]$ correspond with lines through the origin in the same plane. More on this can be found in section **3** below.

The pair of states $[\psi_0]$, $[\psi_1]$ determined by the eigenfunctions are, in the case of 2-level systems, the only stationary states. The state $[\psi_0]$ is the *ground state* and $[\psi_1]$ is the (unique) *excited stationary state.* A state is *complete* if $a_0 \neq 0$ and $a_1 \neq 0$; for complete states $[\psi] = [a_0\psi_0 + a_1\psi_1]$ and

$[\phi] = [b_0\phi_0 + b_1\phi_1]$ their equality, $[\psi] = [\phi]$, is equivalent to the equality of coefficient quotients: $a_0/a_1 = b_0/b_1$ or $a_1/a_0 = b_1/b_0$. This means that there is only one degree of freedom in the space of states of a 2-level system. Note that states in a 2-level system are either stationary or complete.

Note that the stationary states $[\psi_0]$ and $[\psi_1]$ are the unique states for the 1-level subsystems of H.

3.- Space of States: The collection of all states is the *space of states*. Since wave functions form a real vector space E it follows that each state is a line through the origin in this vector space. The collection of all these lines is therefore identical with the space of states.

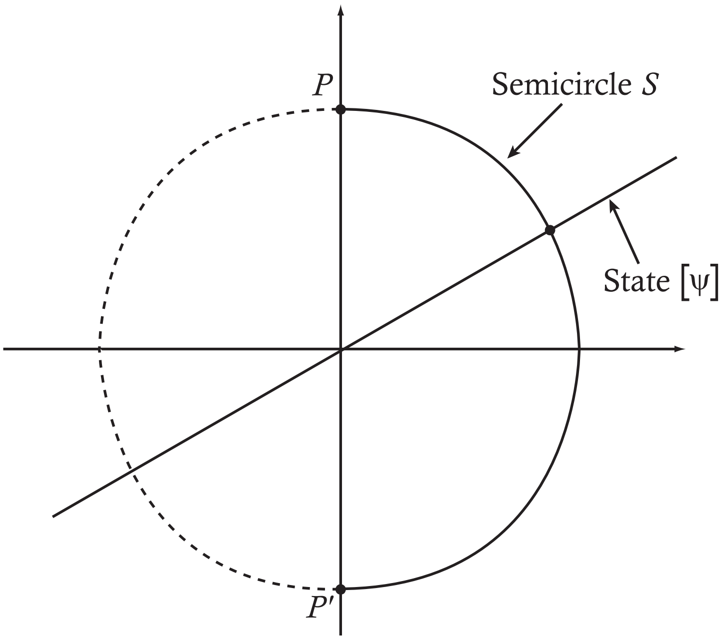

Figure 2. *Each line through the origin corresponds with a unique point in the semicircle except for* $[\psi_1]$ *that determines a pair of points* P, P'.

Readers familiar with the concept will recognize this space of states as the *real projective space* associated to E. From the viewpoint of the description of the physical process projective spaces automatically renormalize all states *without dividing by the norm of the state.* For 2-level systems states correspond with lines through the origin in the a_0a_1-plane. Each line determines a unique point in the semicircle S, except for a redundancy, illustrated in Figure 2, that arises from the fact that the vertical line determines a pair of diametrically opposite points P, P', which are the end points of S. Therefore if these two points are identified to each other a closed curve equivalent to a circle is obtained. The points of this circle correspond exactly with lines that pass through the origin so that the actual space of states *is* a circle. When obtained in this way the circle is known as the *real projective line.* See Figure 3 below.

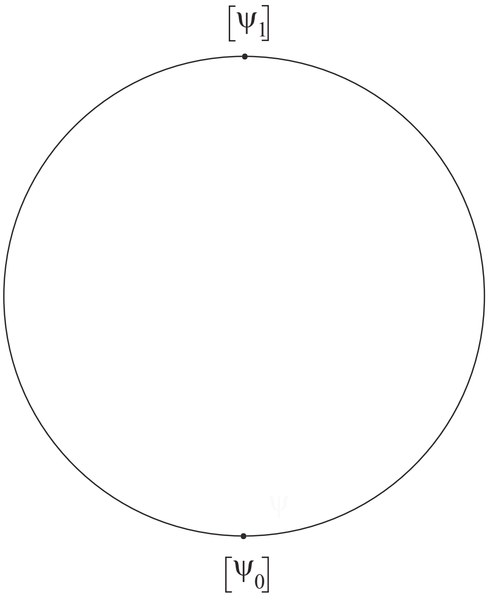

Figure 3. *Space of states of a 2-level system. Each point in this circle uniquely corresponds with a line in the plane. Compare with* Figure 2.

As a consequence a function of state is just a function defined in the real projective line. Also, a time dependent state is a collection of states parameterized by time t and therefore is the same than a parameterized curve contained in the real projective line. Conversely, points in the circle (which are points in a real projective line) are the same than lines through the origin. The pair of diametrically opposite points $[\psi_0]$ and $[\psi_1]$ are the coordinate axes in the $a_0\, a_1$plane. Functions defined on the circle are equivalent to functions defined on non-zero vectors of the plane which are constant along lines passing through the origin (i.e. independent of the scalar factor μ). And finally, a curve parameterized by time and contained in the circle is a collection of lines contained in the plane and similarly parameterized by time.

4.- COORDINATES: Consider the collection V_0 of all 2-level states $[\psi] = [a_0\psi_0 + a_1\psi_1]$ such that $a_0 \neq 0$; they are states with non-zero contribution from the eigenfunction ψ_0. All states except the stationary state $[\psi_1]$ belong to V_0. The scalar valued function f_0 with domain V_0 and defined by the expression

$$f_0([a_0\psi_0 + a_1\psi_1]) = a_1/a_0 \tag{4}$$

is the *projective coordinate system* centered at $[\psi_0]$ in the space of 2-level states. This coordinate system is a one-to-one correspondence between states $[\psi]$ belonging to V_0 and scalars. f_0 transforms the stationary state $[\psi_0]$ into the scalar 0 and states (lines) close to $[\psi_0]$ into scalars close to $f_0([\psi_0]) = 0$. The inverse of f_0 is the function g_0 from scalars to states defined by

$$g_0(x) = [\psi_0 + x\psi_1] \tag{5}$$

The coordinate system can be described pictorially as follows. V_0 consists of all states excluding $[\psi_1]$. Let L_0 be the line $a_0 = 1$. The state $[\psi] = [a_0\psi_0 + a_1\psi_1]$ intersects L_0 at the point $(1, a_1/a_0)$. Therefore $f_0([\psi])$ is the (ordinate of the) intersection of $[\psi]$ and L_0. See Figure 4. Note that the line L_0 is not a state (does not contains the origin). Instead L_0 contains a unique wave function $\psi_0 + (a_1/a_0)\psi_1$ for each state $[a_0\psi_0 + a_1\psi_1]$ in V_0. The inverse g_0 transforms x into the state $g_0(x) = [\psi_0 + x\psi_1]$ =line through $(1, x)$ and the origin.

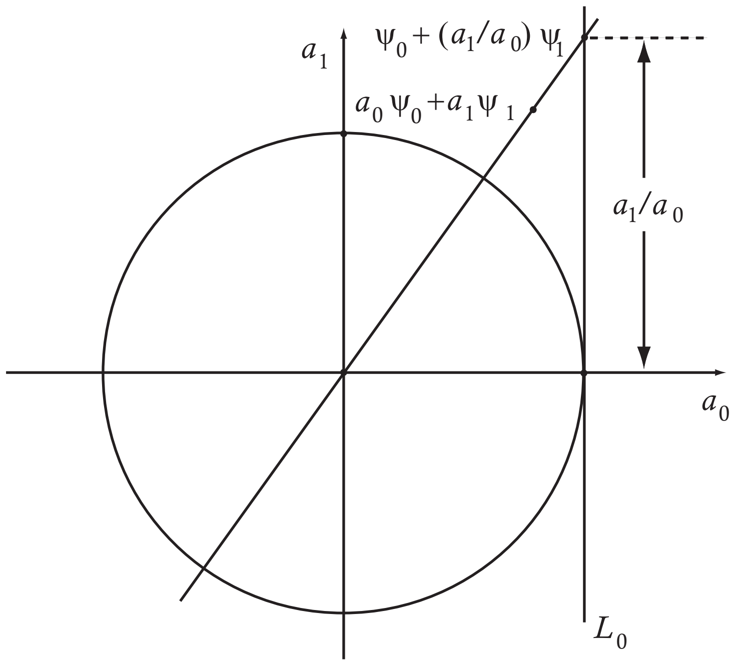

Figure 4. *Coordinates centered at* ψ_0. *Each state* $[\psi] = [a_0\psi_0 + a_1\psi_1]$ *with* $a_0 \neq 0$ *intersects line* L_0 *at a point with ordinate* $a_1/a_0 = f_0([\psi])$.

Similarly, let V_1 consist of all states $[\psi] = [a_0\psi_0 + a_1\psi_1]$ with $a_1 \neq 0$ and define the projective coordinate system f_1 centered at $[\psi_1]$ as

$$f_1([a_0\psi_0 + a_1\psi_1]) = a_0/a_1 \tag{6}$$

The function f_1 is a one-to-one correspondence from V_1 to scalars with inverse $g_1(x) = [x\psi_0 + \psi_1]$. f_1 transforms the stationary state $[\psi_1]$ into the scalar 0 and states near $[\psi_1]$ into scalars near 0. If L_1 is the line $a_1 = 1$ then $f_1([\psi])$ is the abscissa of the intersection of $[\psi]$ with L_1. See Figure 5. Line L_1 contains for each state $[\psi]$ in V_1 a unique wave function $(a_0/a_1)\psi_0 + \psi_1$.

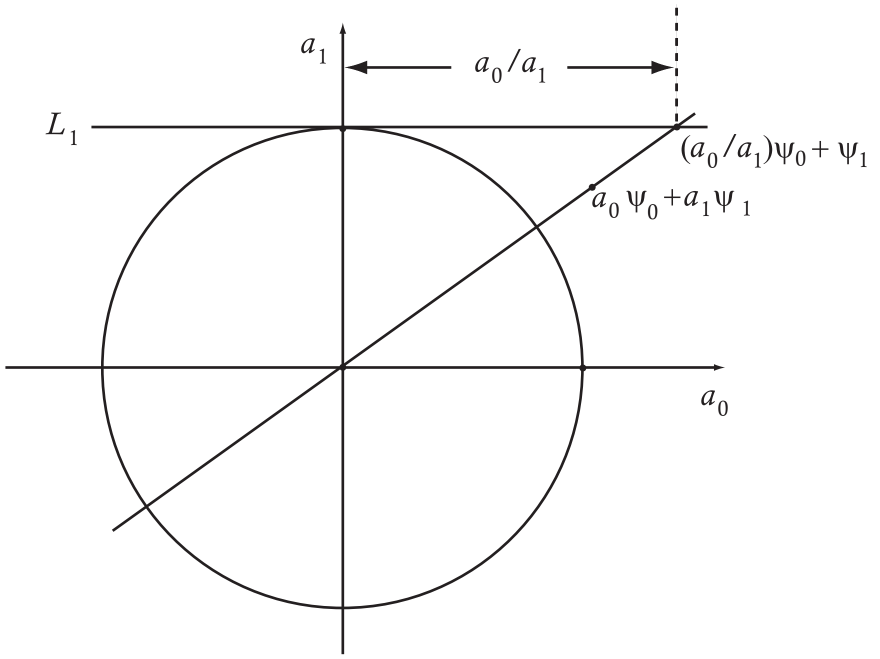

Figure 5. *Coordinates centered at* ψ_1. *Each state* $[\psi] = [a_0\psi_0 + a_1\psi_1]$ *with* $a_1 \neq 0$ *intersects line* L_1 *at a point with abscissa* $a_0/a_1 = f_1([\psi])$.

Projective coordinate systems are *partially defined* functions of state. f_0 is defined in all the space of states except at $[\psi_1]$ and similarly f_1 is defined except at $[\psi_0]$. Complete states belong to both domains V_0 and V_1 and they have well defined images by both f_0 and f_1. Note also that projective coordinates are defined by means of intersections of states with lines L_0 and L_1, while the correspondence of states with points in the projective line defined in section **3** was obtained intersecting states with a semicircle.

5.- APPROXIMATING STATIONARY STATES: The notion of nearness to a stationary state is related to the projective coordinate systems. The 2-level state $[\psi] = [a_0\psi_0 + a_1\psi_1]$ in V_0 (that is, with $a_0 \neq 0$) is *near* the stationary state $[\psi_0]$, denoted $[\psi] \sim [\psi_0]$, if a_1/a_0 is near 0. In other words, $[\psi]$ is near $[\psi_0]$ if $a_0 \neq 0$ and the projective coordinate system centered at $[\psi_0]$ transforms $[\psi]$ in a scalar close to 0. See Figure 4. This can be abbreviated as follows: $[\psi] \sim [\psi_0]$ if $f_0([\psi]) \sim f_0([\psi_0]) = 0$. In a similar way, state $[\psi]$ is near the stationary state $[\psi_1]$ if $a_1 \neq 0$ and $f_1([\psi]) = a_0/a_1$ is near 0, or in abbreviated form: $[\psi_1]$ if $f_1([\psi]) \sim f_1([\psi_1]) = 0$.

Similarly, if $a_1 \neq 0$ then $[\psi] = [(a_0/a_1)\psi_0 + \psi_1]$ and if $|a_0/a_1|$ is a small quantity then $[\psi] \sim [\psi_1]$. See Figure 5.

The definition of nearness to a stationary state depends only on the quotient of coordinates and therefore nearness depends on the state and not on the particular wave function.

If $|a_1/a_0|$ is small then its reciprocal $|a_0/a_1|$ is large, and conversely. Therefore no state can simultaneously approach both stationary states $[\psi_0]$ and $[\psi_1]$. At the other extreme the state $[\psi] = [\psi_0 + \psi_1]$ $(a_0 = a_1 = 1)$ is away from both stationary states.

6.- ENERGY: The second fundamental principle of realism is the definition of energy in terms of the Hamiltonian operator and wave functions.

> ENERGY OF STATES: If the Hamiltonian operator is H then the energy of state $[\psi]$ is given by
>
> $$e_H \quad = e_H([\psi]) = \frac{\langle H(\psi), \psi\rangle}{\langle \psi, \psi\rangle} \tag{7}$$

Replacing ψ with $\mu\psi$ gives the same energy, therefore e_H is a well defined function of state. With this second principle the natural definition of energy as a scalar valued function of state has been recovered. The quantum mechanical definition of the energy observable as a self-adjoint operator, that is, as a special type of wave function valued function of wave functions, is too artificial.

In a two-level system states $[\psi] = [a_0\psi_0 + a_1\psi_1]$ have energy

$$\begin{aligned} e_H &= \frac{\langle H(a_0\psi_0 + a_1\psi_1), a_0\psi_0 + a_1\psi_1\rangle}{\langle a_0\psi_0 + a_1\psi_1, a_0\psi_0 + a_1\psi_1\rangle} && (8) \\ &= -\frac{\lambda_0 a_0^2 + \lambda_1 a_1^2}{a_0^2 + a_1^2} && (9) \end{aligned}$$

which for complete states can also be expressed as

$$\begin{aligned} e_H &= -\frac{\lambda_0 + (a_1/a_0)^2\lambda_1}{1 + (a_1/a_0)^2} \qquad (10) \\ &= -\frac{(a_0/a_1)^2\lambda_0 + \lambda_1}{(a_0/a_1)^2 + 1} \qquad (11) \end{aligned}$$

Taking $x = a_1/a_0$ and $x = a_0/a_1$ the formulas for energy in local coordinates are

$$e_H(g_0(x)) = -\frac{\lambda_0 + \lambda_1 x^2}{1 + x^2} \qquad (12)$$

and

$$e_H(g_1(x)) = -\frac{\lambda_0 x^2 + \lambda_1}{1 + x^2} \qquad (13)$$

See Figure 6.

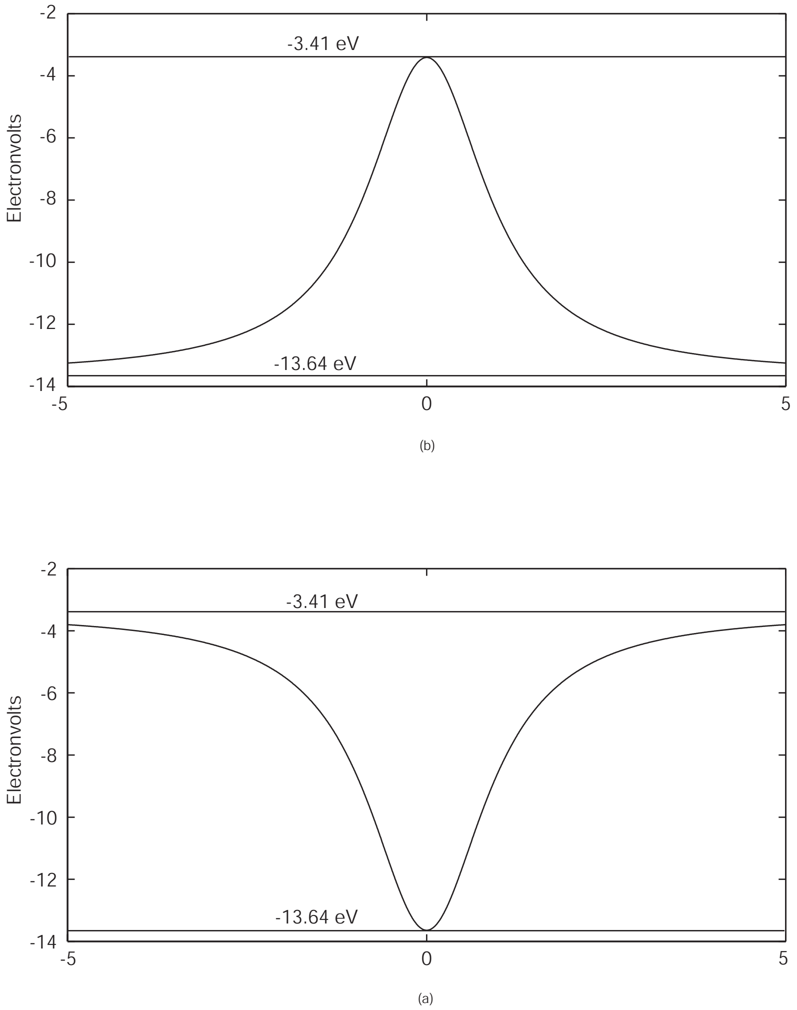

Figure 6. *Energy in local coordinates. Graphs of* (12) *and* (13) *are shown in* (a) *and* (b) *respectively.*

It is immediate from the definitions that the energies of the stationary states are $e_H[\psi_0] = -\lambda_0$ and $e_H[\psi_1] = -\lambda_1$, which are the minimum and maximum of e_H. Formulas (10) and (11) show that $[\psi] \sim [\psi_0]$ implies $e_H([\psi]) \sim -\lambda_0$, and $[\psi] \sim [\psi_1]$ implies $e_H([\psi]) \sim -\lambda_1$. On the other hand for the state with $a_0 = a_1 = 1$ the energy is the average value $e_H = -(1/2)(\lambda_0 + \lambda_1)$.

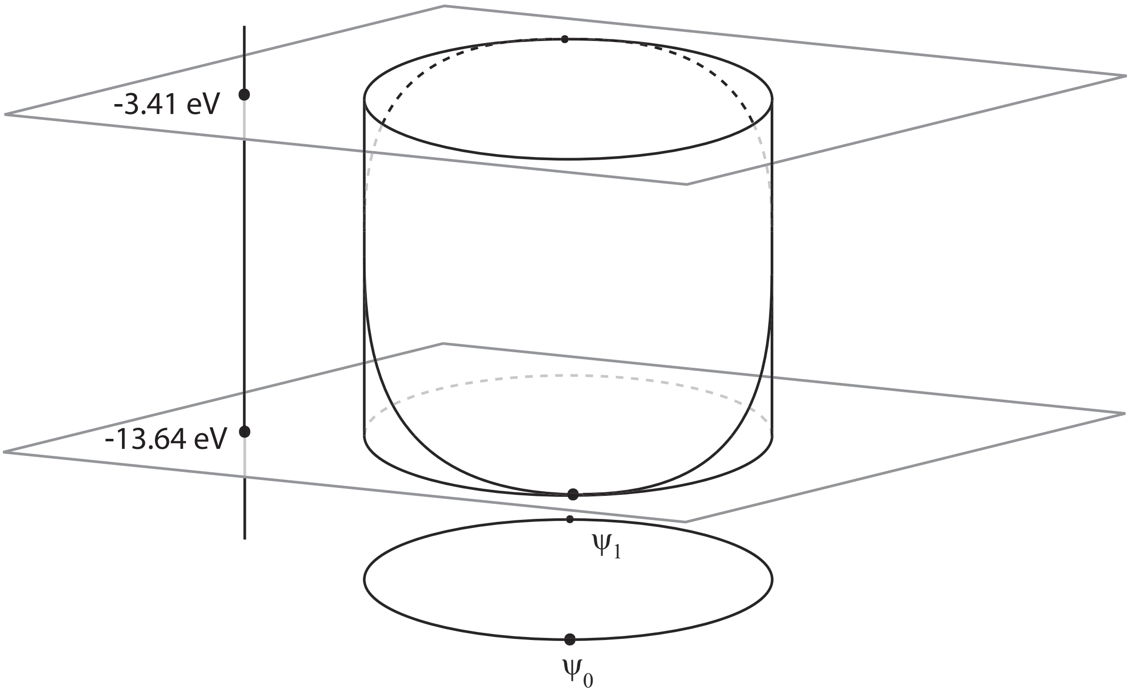

Figure 7. *Graph of the energy as a function of state. Numerical eigenvalues* $-\lambda_0 = -13.64$ *and* $-\lambda_1 = -3.41$ *correspond to energies of the ground state and first excited state of the hydrogen atom.*

The space of states for a 2-level system is a circle, thus the energy is a real valued function of defined in this circle. The graph of this function is a curve contained in a cylinder as shown in Figure 7.

7.- Evolution of Wave Functions: Let h =Planck constant and define $\kappa = 2/h$. Energies and emission times typical for the hydrogen atom are conveniently measured in electronvolts and femtoseconds. In these units $h = 4.13\,\mathrm{eV}\times\mathrm{fs}$ and $\kappa = 0.48\,\mathrm{eV}^{-1}\times\,\mathrm{fs}^{-1}$. The constant κ appears in the real evolution equation.

Given a Hamiltonian operator H one can consider $-\kappa H$ and the time dependent exponential $e^{-\kappa H t}$ which are also operators and can be applied to wave functions in formulas like $-\kappa H(\psi)$ and particularly in the following expression that defines, for a given initial wave function ψ, its time evolution:

$$\psi^{(t)} = e^{-\kappa H t}(\psi) \tag{14}$$

The evolution of wave functions satisfies the *time shift* relation $(\psi^{(s)})^{(t)} = \psi^{(s+t)}$. A calculation shows that the time derivative of $\psi^{(t)}$ is

$$\frac{d\psi^{(t)}}{dt} = \dot{\psi}^{(t)} = -\kappa H e^{-\kappa H t}(\psi) = -\kappa H(\psi^{(t)}) \tag{15}$$

From the way exponential operators are defined it can be proved that if $-\lambda_i$ is an eigenvalue of H belonging to the eigenfunction ψ_i then $e^{-\kappa H t}(\psi_i) = e^{\kappa \lambda_i t}(\psi_i)$ and therefore $e^{\kappa \lambda_i t}$ is an eigenvalue of $e^{\kappa H t}$ belonging to the same eigenfunction ψ_i.

In 2-level systems the Hamiltonian H is a diagonal matrix with respect to the basis ψ_0, ψ_1 implying

$$-\kappa H = \begin{bmatrix} \kappa\lambda_0 & 0 \\ 0 & \kappa\lambda_1 \end{bmatrix} \tag{16}$$

and

$$e^{-\kappa H t} = \begin{bmatrix} e^{\kappa\lambda_0 t} & 0 \\ 0 & e^{\kappa\lambda_1 t} \end{bmatrix} \tag{17}$$

with respect to the same basis. This exponential matrix can be applied to any wave function $\psi = a_0\psi_0 + a_1\psi_1$ resulting in

$$\psi^{(t)} = a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1 \tag{18}$$

The time derivative of the wave function is

$$\dot{\psi}^{(t)} = \kappa(a_0\lambda_0 e^{\kappa\lambda_0 t}\psi_0 + a_1\lambda_1 e^{\kappa\lambda_1 t}\psi_1) \tag{19}$$

From this it follows that if wave functions are represented as points in a plane having rectangular coordinates a_0 and a_1 then the trajectories correspond to solutions of the linear system $(\dot{a_0}, \dot{a_1}) = (\kappa\lambda_0\, a_0, \kappa\lambda_1\, a_1)$. See Figure 8. For t large *negative* the state $\psi^{(t)}$ is a point close to the origin; however, since $\kappa\lambda_0 t < \kappa\lambda_1 t$ the state $[\psi^{(t)}]$ is close to the vertical axis, that is, close to the stationary state ψ_1. At time $t = 0$ one obtains the initial state $\psi^{(0)}$. For t large *positive*, $\psi^{(t)}$ diverges towards infinity, but since $\kappa\lambda_0 t > \kappa\lambda_1 t$ the state $[\psi^{(t)}]$ is close to the horizontal axis, that is, close to the stationary state ψ_0. See Figure 8 below.

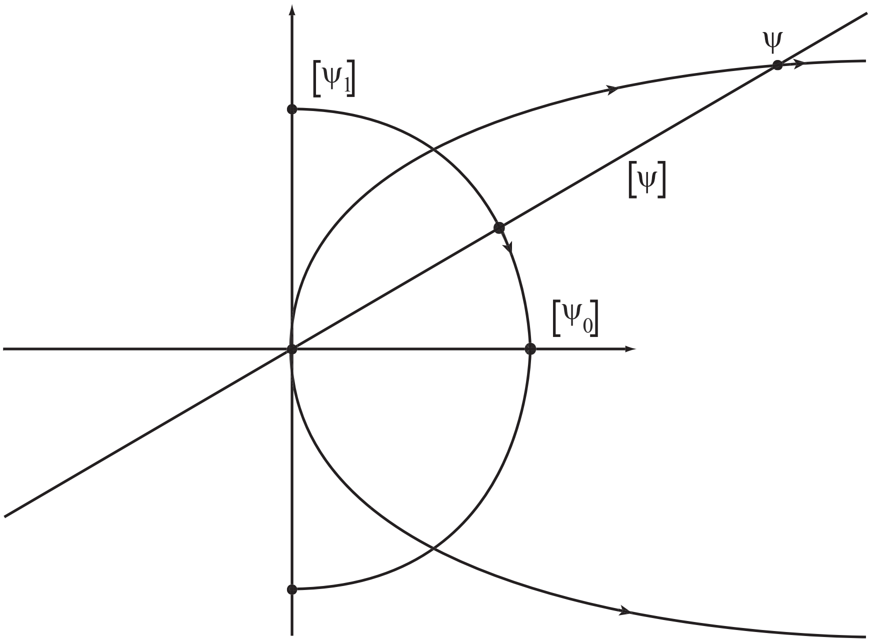

Figure 8. *Trajectories of wave function evolution. A initial point* ψ *in the plane moves as* $\psi^{(t)}$ *along solutions of a linear differential equation and determines a moving line* $[\psi^{(t)}]$ *that pivots around the origin.*

8.- EVOLUTION OF STATES: The careful distinction between wave functions and states implies a not less relevant distinction between evolution of waves and evolution of states. The third basic principle of Realism is the evolution law that specifies in terms the Hamiltonian operator and wave functions the manner in which states move.

EVOLUTION EQUATION: If at time 0 the state of a system is $[\psi]$

then the state at time t is

$$[\psi]^{(t)} = [\psi^{(t)}] = [e^{-\kappa Ht}(\psi)] \tag{20}$$

It must be emphasized that it is the evolution of states, and *not* the evolution of wave functions, the one that describes the physical process. For example, an eigenfunction ψ_i becomes, after time t, equal to the wave function $e^{\kappa\lambda_i t}\,\psi_i$ therefore $e^{\kappa\lambda_i t}(\psi_i) \neq \psi_i$ except at $t = 0$. Clearly then, eigenfunctions *do* change under the evolution. On the contrary, for the stationary state $[\psi_i]$ one has $[\psi_i]^{(t)} = [e^{\kappa\lambda_i t}(\psi_i)] = [\psi_i]$; the last equality follows from the Proportionality Principle. Therefore stationary states (corresponding to eigenfunctions) remain fixed during evolution. They are truly motionless states that do not change in time.

The time shift relation for evolution of states is $([\psi]^{(s)})^{(t)} = [\psi]^{(s+t)}$; this is useful to reduce the study of 2-level trajectories with arbitrary complete initial state to the cases $a_0 = a_1 = 1$ and $a_0 = -a_1 = 1$.

The evolution of states in a 2-level system is

$$[\psi]^{(t)} = [a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1] \tag{21}$$

which according to the Proportionality Principle is equivalent to

$$[\psi]^{(t)} \quad = \quad [\psi_0 + (a_1/a_0)e^{\kappa(\lambda_1-\lambda_0)t}\psi_1] \tag{22}$$

and to

$$[\psi]^{(t)} \quad = \quad [(a_0/a_1)e^{\kappa(\lambda_0-\lambda_1)t}\psi_0 + \psi_1] \tag{23}$$

whenever the required conditions $a_0 \neq 0$ or $a_1 \neq 0$ are satisfied.

Since the space of states is a circle the time dependent state $[\psi]^{(t)}$ is a trajectory in this circle. See Figure 9 and compare with Figure 8. For a stationary initial state the trajectory reduces to a constant. For a complete initial state $[\psi] = [a_0\psi_0 + a_1\psi_1]$ the resulting trajectory comprises a semicircle joining the excited stationary state $[\psi_1]$ to the ground state $[\psi_0]$. Which one of the two possible semicircles depends on the sign of the quotient a_0/a_1. For t large negative, (23) and $\lambda_0 - \lambda_1 > 0$ implies $[\psi^{(t)}] \sim [\psi_1]$. At time $t = 0$ the initial state $\psi = \psi^{(0)}$ is obtained. For t large positive, (22) and $\lambda_1 - \lambda_0 < 0$ implies $[\psi^{(t)}] \sim [\psi_0]$. The *transition energy* of a complete 2-level state is the difference $\lambda_0 - \lambda_1$.

Remark: In concrete physical situations appropriate energy eigenvalues and eigenfunctions are understood. Note that in actual laboratory practice isolation of an excited stationary state, say of the hydrogen atom, is a difficult task that can be achieved for a limited time; therefore the state is perturbed and its motionless condition is lost.

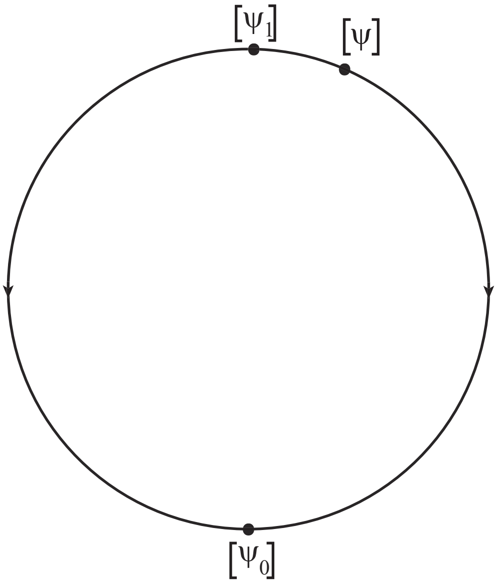

Figure 9. *Trajectories of states. Points in the projective line move as indicated by the arrows. Eigenstates are fixed points.*

9.- NUMERICAL SIMULATION OF TRANSITIONS: Assume that the wave functions are functions of a three dimensional vector argument $\vec{r} = (x, y, z)$, so that $\psi = \psi(\vec{r})$. Then the states $[\psi]$ can be represented as normalized signed densities $\psi/\sqrt{\langle\psi,\psi\rangle}$ defined over the space of three dimensional vectors $\vec{r}$. The time dependent normalized density $\psi^{(t)}/\sqrt{\langle\psi^{(t)},\psi^{(t)}\rangle}$ depicts an actual continuous and deterministic evolution in three space which for $t_a \leq t \leq t_b$ realistically represents the evolution from state $[\psi^{(t_a)}]$ to state $[\psi^{(t_b)}]$. Since densities are signed (can take positive and negative values) the three dimensional plots of the densities should be colored. To visualize a transition of states in ordinary three dimensional space corresponding to the first line in Lymann series, the following specific numerical data can be used. Let time t be femtoseconds varying in $-2 \leq t \leq 2$ with increments of, say, 0.05; $a_0 = a_1 = 1$; $\beta = 0.051$ (Bohr's radius in nanometers); κ, $-\lambda_0$ and $-\lambda_1$ as in Figure 11. For $\vec{r} = (x, y, z)$ take $r = \sqrt{x^2 + y^2 + z^2}$, $\psi_0(\vec{r}) = e^{-r/\beta}$ and $\psi_1(\vec{r}) = (2\beta - r)e^{-r/2\beta}$. Let x, y and z each vary in the interval $[-0.4, 0.4]$ with increments of 0.05; these values are nanometers. A final expression for the time dependent normalized density is

$$\psi^{(t)}(\vec{r}) = \frac{e^{\kappa\lambda_0 t-(r/\beta)} + (2\beta - r)\, e^{\kappa\lambda_1 t-(r/2\beta)}}{\sqrt{e^{2\,\kappa\lambda_0 t-(2r/\beta)} + (2\beta - r)^2\, e^{2\,\kappa\lambda_1 t-(\beta/r)}}} \tag{24}$$

In this particular example densities do not change sign.

10.- EVOLUTION AND PROJECTIVE COORDINATES: The evolution of 2-level states can be described using projective coordinate systems as trajectory of a point in a line. Consider the domain V_0 of f_0 and a complete initial state $[\psi] = [a_0\psi_0 + a_1\psi_1]$. Since $e^{\kappa\lambda_0 t}$ and $e^{\kappa\lambda_1 t}$ are non-zero for all t the time evolution $[\psi]^{(t)} = [a_0 e^{\kappa\lambda_0 t}\psi_0 + e^{\kappa\lambda_1 t}\psi_1]$ is also complete and is in V_0. The function f_0 transforms $[\psi]^{(t)}$ into

$$f_0([\psi]^{(t)}]) = (a_1/a_0)e^{\kappa(\lambda_1-\lambda_0)t} \tag{25}$$

Compare with formula (22). This is the trajectory of the one-dimensional linear system $\dot{x} = (\lambda_1 - \lambda_0)x$ with initial condition a_1/a_0. Therefore when transformed by the projective coordinate f_0 the evolution of states in a 2-level system becomes a 1-dimensional linear evolution. Since $\lambda_1 - \lambda_0 < 0$ trajectories of (25) approach the origin; see Figure 10 (a) and compare also with Figures 8 and 4.

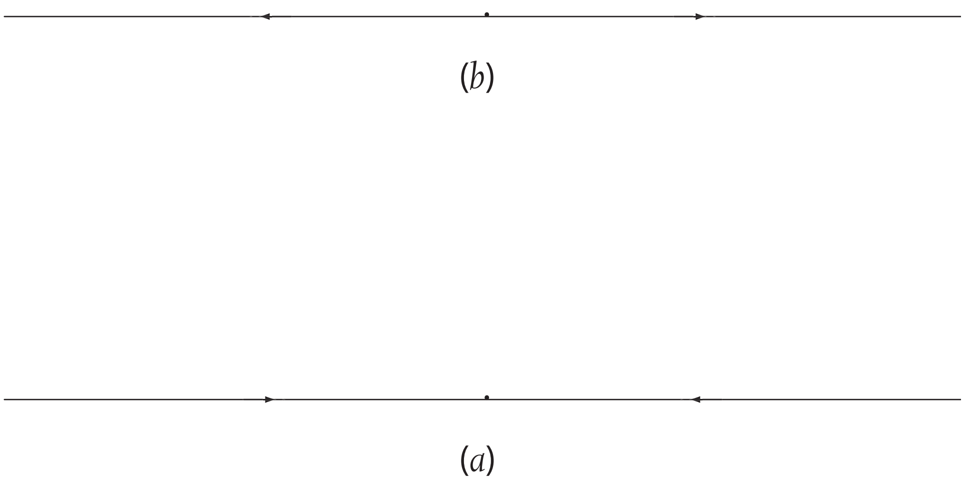

Figure 10. *Evolution in local coordinates is linear. Trajectories of system* (25) *are shown above in* (*a*) *and trajectories of system* (26) *are shown in* (*b*).

Transforming the given complete initial state by means of the projective coordinate system f_1 one obtains

$$f_1([\psi]^{(t)}]) = (a_0/a_1)e^{\kappa(\lambda_0-\lambda_1)t} \tag{26}$$

which is a trajectory of the one-dimensional linear system $\dot{x} = (\lambda_0 - \lambda_1)x$ with initial condition a_0/a_1. Again, the evolution in the non-linear space of 2-level systems transforms *via* the projective coordinate system centered at $[\psi_1]$ into a linear evolution on the real line. In this case $\lambda_0 - \lambda_1 > 0$ and the linear trajectories are as shown in Figure 10 (b). Compare with Figures 8 and 5. From the viewpoint of the inverses of the coordinates, the 1-dimensional linear evolution equations (25) and (26) are fitted together using the inverses g_0 and g_1 and provide on the non-linear space of 2-level states a globally well defined non-linear evolution.

The three principles of Realism can be used to further clarify some details about electron transitions and energy radiation. This is the topic of the sections ahead.

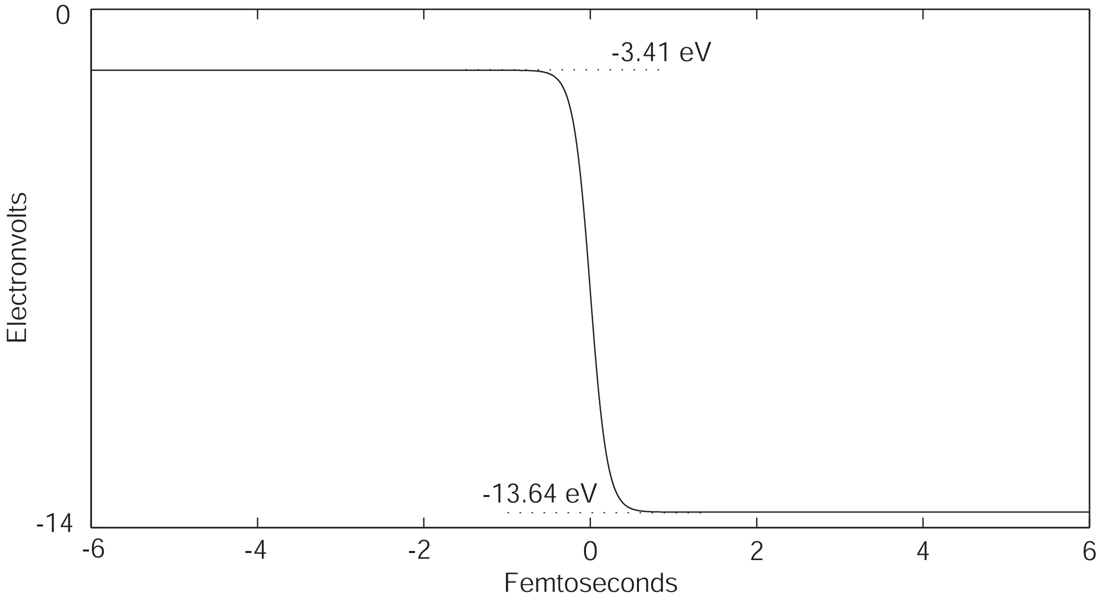

Figure 11. *Energy as function of time for the first line in Lymann series: This is formula 34 with energy given in electronvolts and time in femtoseconds. Here* $\kappa = 0.48\ \mathrm{eV}^{-1}\times\mathrm{fs}^{-1}$. *The initial and final energies are* $-\lambda_1 = -3.41$ eV *and* $-\lambda_0 = -13.64$ eV.

11.- EVOLUTION OF ENERGY: A system with initial state $[\psi]$ becomes, after time t, the state $[\psi]^{(t)} = [e^{-\kappa H t}(\psi)]$ and therefore the energy of the system at this time is

$$e_H(t) = e_H([\psi]^{(t)}) = \frac{\langle H(\psi^{(t)}), \psi^{(t)}\rangle}{\langle \psi^{(t)}, \psi^{(t)}\rangle} \tag{27}$$

For an initial stationary state $[\psi_i]$ with eigenvalue $-\lambda_i$ one has $e_H = -\lambda_i$ at all times t. From the expression (27) for the energy as a function of time, the rate of change of the energy $e_H(t)$ can be calculated for any initial state $[\psi]$ as

$$\frac{d\, e_H(t)}{dt} = -2\kappa \left(\frac{\langle H^2(\psi^{(t)}), \psi^{(t)}\rangle}{\langle \psi^{(t)}, \psi^{(t)}\rangle} - \frac{\langle H(\psi^{(t)}), \psi^{(t)}\rangle^2}{\langle \psi^{(t)}, \psi^{(t)}\rangle^2} \right) \tag{28}$$

$$= -2\kappa (e_{H^2}[\psi]^{(t)} - e_H^2[\psi]^{(t)}) \tag{29}$$

According to Schwartz inequality, inner products satisfy $\langle w, w\rangle\langle v, v\rangle \geq \langle w, v\rangle^2$ and if $v \neq 0$ one can write this as $(\langle w, w\rangle/\langle v, v\rangle) - (\langle w, v\rangle/\langle v, v\rangle)^2 \geq 0$ which for $w = H(\psi^{(t)})$ and $v = \psi^{(t)}$ gives $e_{H^2}[\psi]^{(t)} - e_H^2[\psi]^{(t)} \geq 0$. If the initial state is an eigenstate then the energy of the system is stationary ($de_H/dt = 0$) and for all other initial states the energy decreases ($de_H/dt < 0$). Compare with Figure 11 above.

For 2-level systems with initial state $[\psi] = [a_0\psi_0 + a_1\psi_1]$ the energy at time t equals

$$\begin{aligned} e_H(t) &= \frac{\langle H(a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1), a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1\rangle}{\langle a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1, a_0 e^{\kappa\lambda_0 t}\psi_0 + a_1 e^{\kappa\lambda_1 t}\psi_1\rangle} \\ &= -\frac{\lambda_0 a_0^2 e^{2\kappa\lambda_0 t} + \lambda_1 a_1^2 e^{2\kappa\lambda_1 t}}{a_0^2 e^{2\kappa\lambda_0 t} + a_1^2 e^{2\kappa\lambda_1 t}} \end{aligned} \tag{30}$$

which for $a_0 \neq 0$ or $a_1 \neq 0$ transforms into

$$e_H(t) = -\frac{\lambda_0 + \lambda_1 (a_1/a_0)^2 e^{2\kappa(\lambda_1-\lambda_0)t}}{1 + (a_1/a_0)^2 e^{2\kappa(\lambda_1-\lambda_0)t}} \tag{31}$$

or

$$e_H(t) = -\frac{\lambda_0 (a_0/a_1)^2 e^{2\kappa(\lambda_0-\lambda_1)t} + \lambda_1}{(a_0/a_1)^2 e^{2\kappa(\lambda_0-\lambda_1)t} + 1} \tag{32}$$

If one takes $x = a_1/a_0$ or $x = a_0/a_1$ expressions for the evolution of energy in the projective coordinate systems are obtained. If the initial state is complete then for t large negative, $\lambda_0 - \lambda_1 > 0$ together with (32) implies $e_H(t) \sim -\lambda_1$; for $t = 0$ one gets $e_H(0)$ =energy of the initial state; and for t large positive, $\lambda_1 - \lambda_0 < 0$ and (31) imply $e_H(t) \sim -\lambda_0$. Equivalently $\lim_{t\to-\infty} e_H(t) = -\lambda_1$ and $\lim_{t\to\infty} e_H(t) = -\lambda_0$. Stationary initial states, themselves fixed in the space of 2-level states, have constant energy

$$e_H[\psi_0](t) = -\lambda_0 \quad \text{and} \quad e_H[\psi_1](t) = -\lambda_1 \tag{33}$$

independently of t.

For the initial state $[\psi_0 + \psi_1]$ formula (30) simplifies to

$$e_H(t) = -\frac{\lambda_0 e^{2\kappa\lambda_0 t} + \lambda_1 e^{2\kappa\lambda_1 t}}{e^{2\kappa\lambda_0 t} + e^{2\kappa\lambda_1 t}} \tag{34}$$

12.- RADIATED ENERGY: Assume that a given initial state $[\psi]$ is such that for large negative times $t < 0$ one has $e_H(t) \sim -\lambda_{init}$; more precisely let $\lim_{t\to-\infty} e_H(t) = -\lambda_{init} < 0$. The natural definition of *radiated energy* for this initial state is

$$r_H(t) = r_H([\psi]^{(t)}) = -\lambda_{init} - e_H(t) = -\lambda_{init} - \frac{\langle H(\psi^{(t)}), \psi^{(t)}\rangle}{\langle \psi^{(t)}, \psi^{(t)}\rangle} \tag{35}$$

It was established in section **11** above that for a 2-level complete initial state $\lim_{t\to-\infty} e_H(t) = -\lambda_1$ and therefore in the 2-level case the radiated energy is

$$r_H(t) = -\lambda_1 + \frac{\lambda_0 a_0^2 e^{2\kappa\lambda_0 t} + \lambda_1 a_1^2 e^{2\kappa\lambda_1 t}}{a_0^2 e^{2\kappa\lambda_0 t} + a_1^2 e^{2\kappa\lambda_1 t}} \tag{36}$$

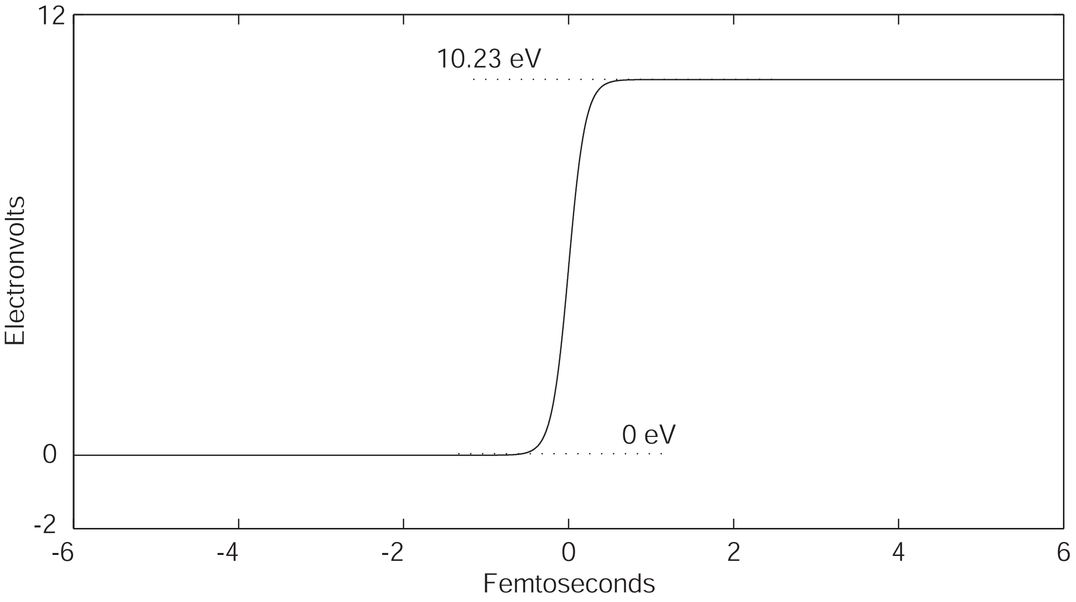

Figure 12. *Radiated energy as function of time for a 2-level system: First line in Lymann series. This is formula 37 with energy given in electronvolts and time in femtoseconds. Values of κ, λ_1 and λ_0 as in Figure 11. The transition energy is* $\lambda_0 - \lambda_1 = 10.23\,\mathrm{eV}$.

so that $r_H(t)$ increases monotonically starting for t large negative at $r_H(t) \sim 0$, and increasing progressively towards $\lambda_0 - \lambda_1$. A simpler expression is obtained for $a_0 = a_1 = 1$, namely

$$r_H(t) = -\lambda_1 + \frac{\lambda_0 e^{2\kappa\lambda_0 t} + \lambda_1 e^{2\kappa\lambda_1 t}}{e^{2\kappa\lambda_0 t} + e^{2\kappa\lambda_1 t}} \tag{37}$$

See Figure 12. The total amount of radiated energy equals the difference

$$\lim_{t\to-\infty} r_H(t) - \lim_{t\to\infty} r_H(t) = \lambda_0 - \lambda_1 \tag{38}$$

The radiated energy $r_H(t)$ is a continuous function of t and takes all from 0 and $\lambda_0 - \lambda_1$. Customary experimental settings usually detect this eigenvalue difference, thus the myth of the 'discontinuous quantum jump'. To obtain intermediate energy values, say $-\lambda$ with $-\lambda_0 < -\lambda < -\lambda_1$, would require to stop the radiation process momentaneously at some state $[\psi]$ with energy $e_H([\psi]) = -\lambda$. But in such case $-\lambda$ would be a stationary energy value and the system would have at least three levels, not two.

13.- POWER: The *power* or *intensity* of the radiated energy for a given initial state $[\psi]$ is, at time t

$$p_H(t) = p_H[\psi]^{(t)} = \frac{dr_H}{dt} = -\frac{de_H}{dt} \tag{39}$$

and from (29) one has

$$p_H(t) = 2\kappa(e_{H^2}[\psi]^{(t)} -^2 [\psi]^{(t)}) \tag{40}$$

In the case of 2-level systems a calculation gives

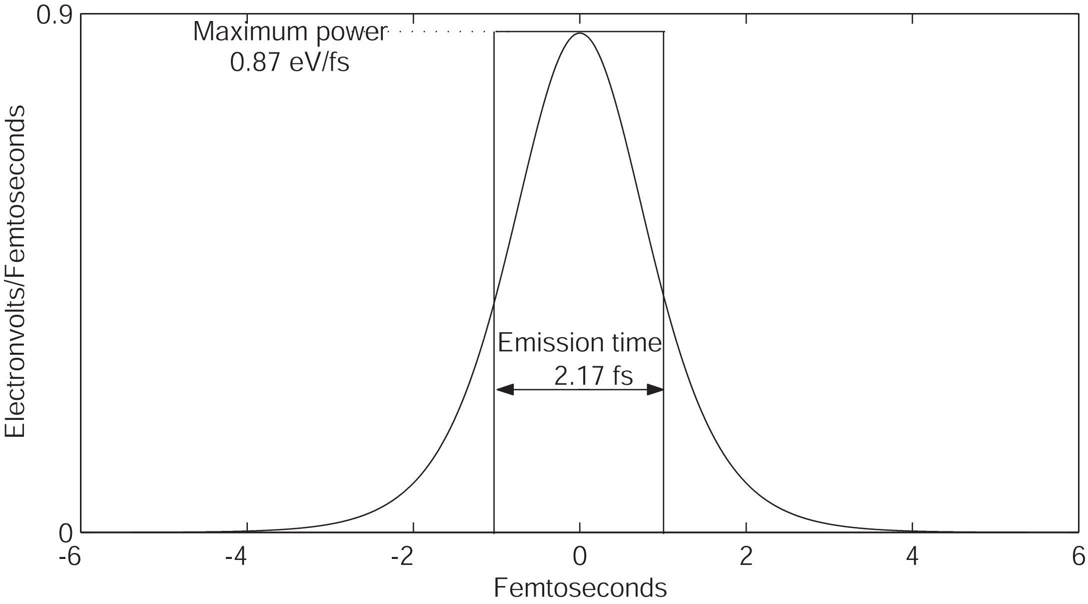

Figure 13. *Radiation power as function of time for first line in Balmer series. This is formula* (42) *with* $\lambda_0 = 3.41$ *and* $\lambda_1 = 1.51$. *The area under the rectangle equals the area under* $p_H(t)$. *A constant pulse of duration* $\delta_H = 2.17$ fs *and amplitude* $p_H^{\max} = 0.87$ eV/fs *delivers as much energy as* $p_H(t)$, *namely, the transition energy* $\lambda_0 - \lambda_1 = 1.9$ eV.

$$p_H(t) = 2\kappa \frac{(\lambda_0 - \lambda_1)^2 \, a_0^2 \, a_1^2 \, e^{2\kappa(\lambda_0+\lambda_1)t}}{(a_0^2 \, e^{2\kappa\lambda_0 t} + a_1^2 \, e^{2\kappa\lambda_1 t})^2} \tag{41}$$

When the initial state is $[\psi_0 + \psi_1]$ the previous formula becomes

$$p_H(t) = 2\kappa \frac{(\lambda_0 - \lambda_1)^2 e^{2\kappa(\lambda_0+\lambda_1)t}}{(e^{2\kappa\lambda_0 t} + e^{2\kappa\lambda_1 t})^2} \tag{42}$$

The power $p_H(t) = p_H[\psi]^{(t)}$ attains, for complete initial 2-level states $[\psi]$, its unique maximum at the *time of maximum radiation power*

$$T^{\max} = T^{\max}([\psi]) = \frac{\ln |a_0/a_1|}{\kappa(\lambda_0 - \lambda_1)} \tag{43}$$

For $a_0 = a_1 = 1$, $T^{\max} = 0$. The corresponding maximum value attained by p_H at $t = T^{\max}$ is

$$p_H^{\max} = p_H(T^{\max}([\psi])) = \frac{\kappa(\lambda_0 - \lambda_1)^2}{2} \tag{44}$$

A crucial property has been established: The maximum radiation intensity does not depend on the initial state and *is proportional to the square of the transition energy*; if the transition energy is doubled then the maximum intensity increases fourfold.

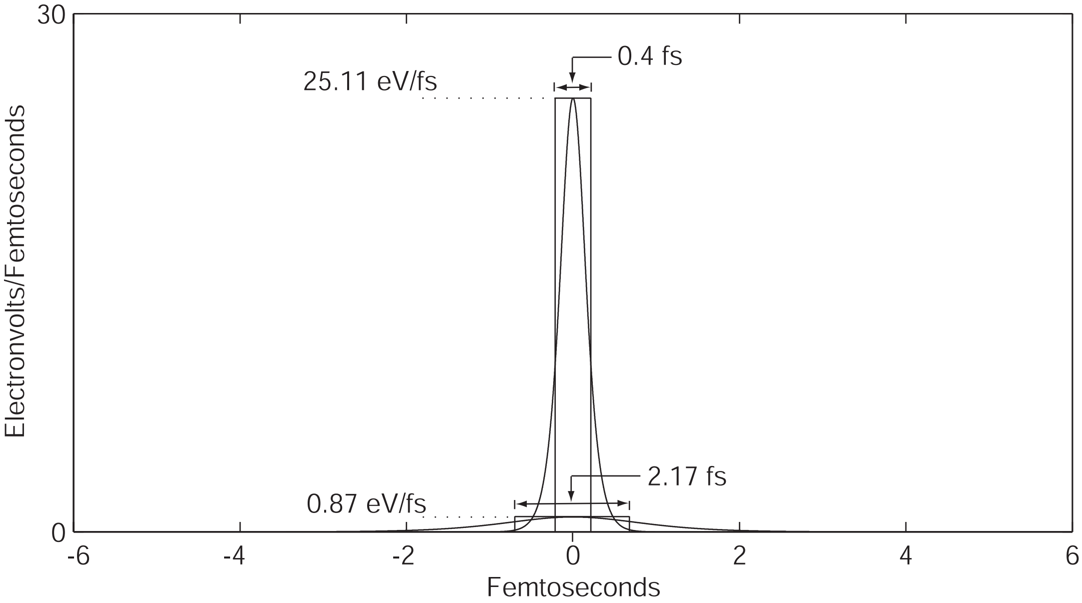

Figure 14. *Comparison of* p_H, $p_H^{\max}$ *and* δ_H *for first lines of Lymann and Balmer series. Vertical scale is different from* Figure 13.

At constant power with amplitude equal to $p_H^{\max}$ the transition energy $\lambda_0 - \lambda_1$ would be radiated in a time interval of length

$$\delta_H = \frac{\lambda_0 - \lambda_1}{p_H^{\max}} = \frac{h}{\lambda_0 - \lambda_1} \tag{45}$$

The quantity δ_H has the dimensions of time and is, by definition, the *emission time* or *duration* of $p_H(t)$. Therefore *the emission time of $p_H(t)$ does not depend on the initial state and is inversely proportional to the transition energy.* If the transition energy is doubled then the emission time is halved. Relations (44) and (45) are consequences of the real evolution equation and have important implications to be explained in the next section. Power, their maxima and emission times for first lines in Lymann and Balmer series are shown in Figure 15.

14.- PHOTONS: Photons are the carriers of the transition energy. With the three stated principles and one additional hypothesis Realism can partially resolve the structure of photons. Assume then the following

> PHOTON HYPOTHESIS: The energy radiated by the system travels in ordinary 3-space as a localized wave moving parallel to a straight line L with constant speed c.

In more detail, choose the initial time $t = 0$ at the instant when the 2-level system has maximum radiation power and choose the x-axis parallel to the propagation axis L. At a later instant $t > 0$, when the system has radiated all its energy, the photon has an energy distribution $P(x, y, z)\, dx\, dy\, dz$ in 3-space such that $\int P(x, y, z)\, dx\, dy\, dz = \lambda_0 - \lambda_1$; this requires only energy conservation. The photon hypothesis then says that this energy propagates in space according to $P(x - ct, y, z)$. The quantity $\Phi_H(x) = \int P\, dy\, dz$ is the energy in the plane N_x normal to L at x. Since $x = ct$ and $dx = c\, dt$ one has $t = x/c$, $dt = dx/c$ and (42) implies

$$\Phi_H(x) = 2\kappa(\lambda_0 - \lambda_1)^2 \frac{e^{2\kappa(\lambda_0+\lambda_1)(x/c)}}{c\left(e^{2\kappa\lambda_0(x/c)} + e^{2\kappa\lambda_1(x/c)}\right)^2} \tag{46}$$

with a unique maximum value given by

$$\Phi_H^{\max} = \frac{p_H^{\max}}{c} = \frac{(\lambda_0 - \lambda_1)^2}{hc} \tag{47}$$

See Figure 15.

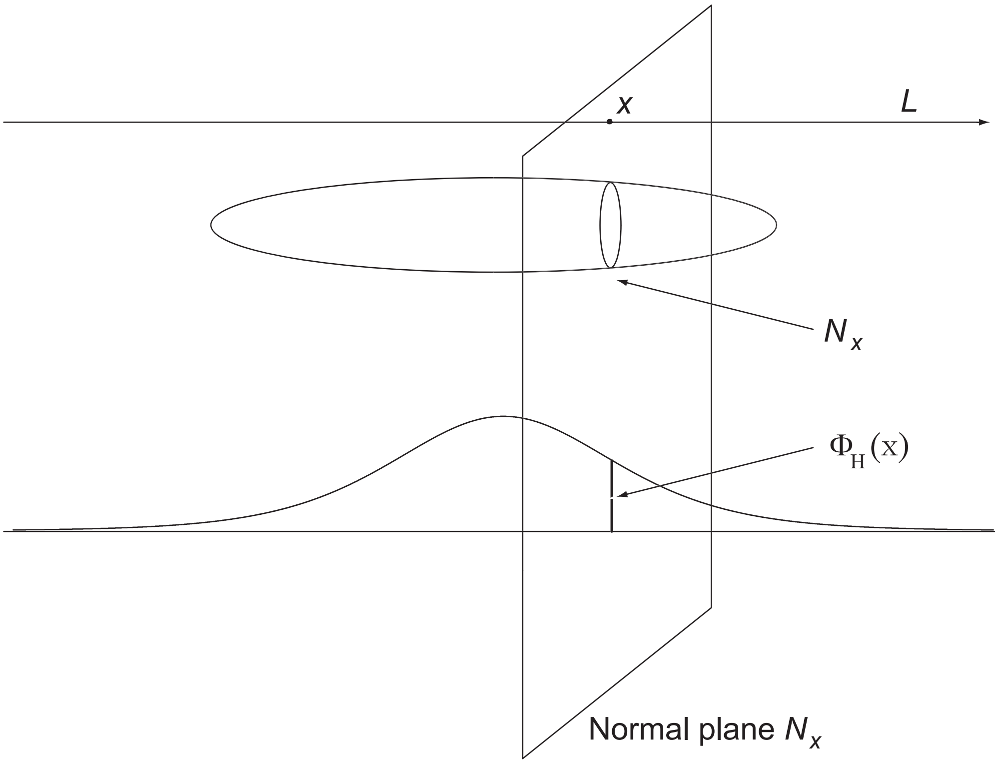

Figure 15. *The energy of the photon contained in the normal plane N_x equals $\Phi_H(x)$. Elongated figure of photon is for illustrative purposes; only $\Phi_H(x)$, not $P(x, y, z)$, is known.*

Again the evolution equation has provided an important relation: *The maximum energy density along the propagation axis is proportional to the square of the transition energy.* Double transition energy implies a fourfold increase in $\Phi_H^{\max}$.

Formula (46) partially resolves the structure of the photon. A more detailed description of the photon, not yet available, would explicitly give $P(x, y, z)$; an even more complete description would be a formula for the photon as a traveling electromagnetic pulse.

The *wavelength* Λ of the photon can now be defined as

$$\Lambda = \frac{1}{\Phi_H^{\max}} \int_{-\infty}^{\infty} \Phi_H(x)\, dx = \frac{(\lambda_0 - \lambda_1)\, h\, c}{(\lambda_0 - \lambda_1)^2} = \frac{h\, c}{\lambda_0 - \lambda_1} \tag{48}$$

Equivalently, Λ is the length traveled at speed c during an emission time interval δ_H:

$$\Lambda = c\delta_H \tag{49}$$

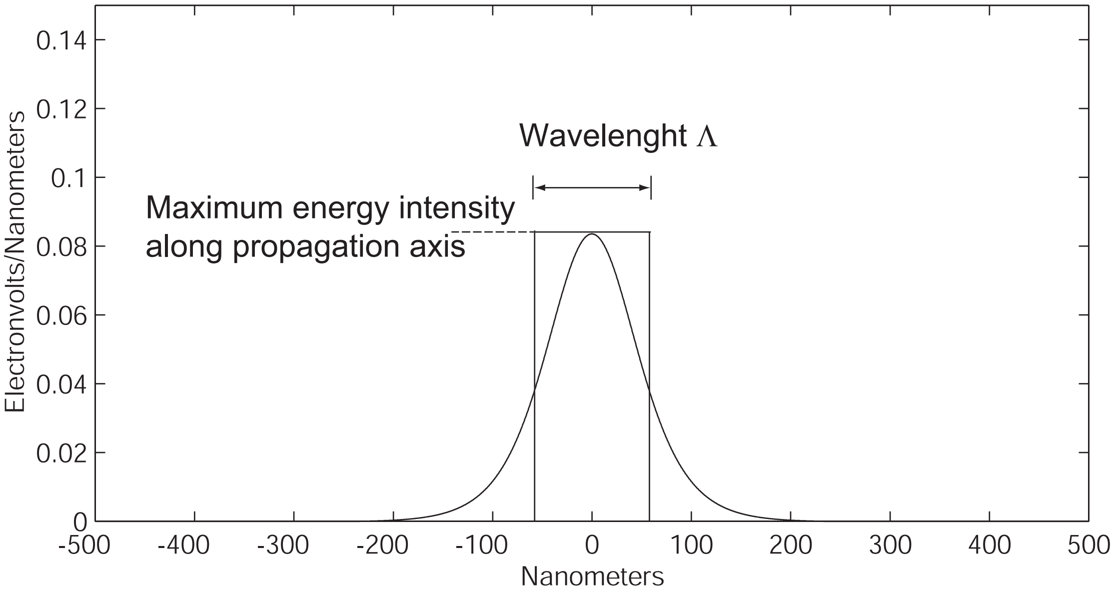

Figure 16. *Energy density along propagation axis for first line in Lymann series. Time given in femtoseconds and distance in nanometers with speed of light* $c = 300$ *nm/fs. Energy density is given in electronvolts over nanometers. The area under the rectangle* = *area under* $\Phi_H(x)$ = *transition energy* $\lambda_0 - \lambda_1 =$ 10.23 eV. *Maximum density* $\Phi_H^{\max} = 0.08$ eV/nm. *Wavelength* $\Lambda = 121\,\mathrm{nm}$.

The definition of wavelength implies

$$\lambda_0 - \lambda_1 = \frac{h\,c}{\Lambda} \tag{50}$$

This is the formula postulated by Einstein, relating the energy and the wavelength of a photon. As shown, Realism deduces the formula from the Photon Hypothesis.

The energy density $\Phi_H(x)\,dx$ does not originate on a periodic or oscillatory movement of the state. Energy radiation occurs as a deterministically defined, continuous process. This process is a trajectory in the space of states. The trajectory begins near the excited stationary state, progresses to intermediate states that are reached and left behind, and asymptotically approaches the ground state. While the system progresses along its trajectory it radiates. The radiation power $p_H(t)$ and the trajectory $[\psi]^{(t)}$ are non-oscillatory. The non-oscillatory nature of the energy radiation is clearly illustrated in Figure 16 and for the trajectory itself see Figure 9. Therefore there is no way to define a physically meaningful frequency ν for $\Phi_H(x)$. Nevertheless Einstein formula has traditionally been expressed as $E = h\nu$.

Summing up, individual photons are electromagnetic pulses with a wavelength and *are not* periodic electromagnetic oscillations with a frequency. When many individual photons are succesively emitted it can be argued that a train of waves exist and in this case a frequency can possibly be defined.

Reasonable as it is, the definition of wavelength for an energy distribution shaped like $\Phi_H(x)$ is, in the last analysis, a *convention* which together with the value $\kappa = 2/h$ for the exponent in the evolution equation 20 implies Einstein formula (50). Had we taken $\kappa = 1/h$ then redefining a wavelength twice longer would still imply 50. But regardless of the exact value of κ Realism establishes that photons with double the energy are emitted in half the emission time and are packed along their propagation axis in half the wavelength. Therefore $\kappa = 2/h$ should be considered provisional and the final decision as to whether κ equals $2/h$, $1/h$ or some other multiple α/h is better decided by experiment. One such experiment could be the direct measurement of $\Phi_H^{\max}$ for a photon of known energy.

In a letter to his friend Besso, Albert Einstein stated: '...anyone that claims to understand $E = h\nu$ is a liar'. Certainly the formula could not be comprehended on the basis of a qualitatively incorrect unitary evolution equation.

15.- INTERPRETATION OS STATES: According to the Proportionality Principle states $[\psi]$ are proportionality classes of wave functions and are therefore dimensionless, as are, for example, angles, or any ratios between physical quantities of the same magnitude. That $[\psi]$ is dimensionless should not be surprising. After all, the Universe has functioned and will function in the foreseeable future independently of man-made systems of units.

The dimensionless nature of $[\psi]$ also means that certain fundamental physical level has been reached. State $[\psi]$ represents an electron without reference to predefined magnitudes. The physical relevance of this construction is contained in the fact that it delinearizes the space of states providing an equally non-linear evolution equation; this non-linear evolution equation accurately predicts one of the most basic of microscopic phenomena, namely, energy radiation. That the specific expressions obtained in the previous sections provide new and consistent physical information as are the formulas for radiated energy, power, duration, wavelength, and others, reflect the correctness of Realism.

END