Projective Dynamical Systems and Realism

(Beta version 1.0, March 31, 1996)

Daniel Crespin
Facultad de Ciencias
Universidad Central de Venezuela

Abstract

Operators H in Hilbert space E define in projective space PE dynamical systems and an observable. Systems defined by unitary and by self adjoint operators are considered. Part I studies the finite dimensional cases and Part II discusses infinite dimensional ones. The paper contains the Projective Spectral theorem, a mathematical result relevant for Realism, a deterministic, continuous, chaotic and non-linear wave theory of atoms based on Schrödinger self-adjoint operator. The PST is to projective dynamics as the linear spectral theorem is to linear dynamics.

Foreword

The original version of this paper was [2], a preprint circulated over a dozen years ago. It was prepared with a typewriter and without the convenience of a word processor. Meanwhile *Realism* has appeared creating a specific interest for that work. Therefore the author decided to reissue that paper and the outcome is this electronic manuscript. The numerous minor modifications have not changed in any essential way the ideas or results contained in the original work and it is hoped that they have enhanced readability.

The author expresses once more his gratitude to Vidal Alonso for many generous hours of discussion and to Emery Dunia and Renato Iraldi for their encouragement. It is a pleasure to aknowledge also the present support from all the staff of Laboratorio de Cómputo Científico del Departamento de Matemáticas, Facultad de Ciencias, UCV, under the leadership of José Rafael González and including Luis Morales, Mireya Morales, José Luis Piñero, Manuel Maia, Leobardo Valera and Alexander Velásquez. Without their help the global Internet distribution of local scientific work would not have been possible.

Part I. Finite Dimensional Projective systems

1.-Outline. This paper deals with mathematical prerequisites for a continuous, deterministic, non-linear and chaotic wave theory of atoms. The physical theory is called *Realism* and is discussed at length in [1]. The purpose of Realism is to explain basic aspects of atomic behaviour in a logical, reasonable and common sense way *without* using the Uncertainty Principle, wave-particle duality, probabilistic interpretation of wave functions, probabilistic reduction of wave packet, discontinuous quantum jumps, hidden variables, virtual particles, and so on. Realism is an alternative to Quantism. The mathematical results and techniques discussed here involve standard tools of Global Analysis including infinite dimensional manifolds, vector fields, flows and Morse theory. These appear in a natural way due to the non-linear personality of Realism. The manifolds of interest are projective spaces and the vector fields and flows are those to be called *projective fields* and *projective flows*. They arise in the following way. The states of a microscopic physical system are commonly represented by elements ψ of a vector space E of wave functions. However, this representation is known to be not completely accurate; it is points in the projective space PE (= space of lines in E or space of *Hilbert rays*) that provide the correct representation, and this is the reason to study these manifolds. Furthermore, the evolution equation of a quantum system is of the form

$$\frac{\partial \psi}{\partial t} = A(\psi) \tag{1}$$

where A is certain linear partial differential operator, usually $A = -iH$, $H = -\nabla^2 + U$ = Schrödinger Hamiltonian operator.

Equation (1) can be considered as a linear vector field on E. The flow of this field, to be denoted $\exp(At)$, is a one-parameter group of linear transformations on E. But it has to be a flow on PE the one that gives the physical evolution. The linear flow induces in a natural way a flow on PE and this is generated by a vector field or section of the tangent bundle of PE; these are the *projective flow* and *projective field* associated to equation (1), to be denoted $[\exp(At)]$ and X_A respectively. See sections 13–15 for precise definitions. So, projective flows and fields are relevant for the study of microscopic physical systems. It will be seen that, although related in a simple way, the

linear and projective flows are rather different in structure.

The main mathematical result presented here is Theorem 1 of section 18. It is a first and typical case of a class of results that can be called a *Projective Spectral Theorem* or *PST*. For diagonalizable linear operators A over the real or complex number fields, Theorem 1 describes the projective flow $[\exp(At)]$ in terms of eigenvalues of A and certain vector bundles constructed from the eigenspaces of A.

The wave theoretical systems used by both Quantism and Realism are infinite dimensional. However, it will be convenient and useful to discuss in the first part of this paper finite dimensional projective systems.

2.-Finite dimensional linear Hamiltonians. The finite dimensional unitary linear evolution will be discussed first. Assume in this section that E^n is an n-dimensional complex Hilbert space, $n \geq 2$, with Hermitian inner product $\langle\ ,\ \rangle$; this defines in a natural way a symplectic structure on E^n. Consider a self-adjoint operator $H : E^n \to E^n$ with simple negative eigenvalues $-\lambda_0 < \cdots < -\lambda_{n-1} < 0$ and corresponding one-dimensional eigenspaces $L_0, \ldots, L_{n-1}$ mutually orthogonal to each other.

Let $y \in E^n$. The vector field $-iH$ or, equivalently, the linear system of ordinary differential equations

$$\frac{dy}{dt} = -iH(y) \tag{2}$$

is a finite dimensional version of the unitary evolution equations of standard quantum mechanics.

A direct calculation shows that $-iH$ is the Hamiltonian field of the function $\frac{1}{2}e(y)$ where

$$e(y) = \langle H(y), y\rangle \tag{3}$$

is the *energy function* of the operator H on E^n. The only critical point of e is the origin. The linear flow $\exp(-iHt)$ splits as a sum of one-dimensional complex flows

$$\exp(-iHt) = \bigoplus_{j=0}^{n-1} \exp(i\lambda_j I_j t) \tag{4}$$

with $I_j : L_j \to L_j$ the identity operator in the complex line L_j. It follows that all trajectories of equation (2) are bounded and lie in tori. In more detail, choose for each j a normalized eigenvector $y_j \in L_j$. The vectors $y_0, \ldots, y_{n-1}$ will then be an orthonormal basis of E^n. Given a state

$$y^{(0)} = \zeta_0^{(0)} y_0 + \cdots + \zeta_{n-1}^{(0)} y_{n-1} \in E^n,\ \zeta_j \in \mathbf{C} \tag{5}$$

the trajectory of $\exp(-iHt)$ with initial condition $y^{(0)}$ is the parametrized curve

$$y^{(t)} = \sum_{j=0}^{n-1} \exp(i\lambda_j t)\zeta_j^{(0)} y_j \tag{6}$$

This is contained in the torus which is the cartesian product of circles C_j, each contained in the complex line L_j, centered at the origin and with radius $r_j = |\zeta_j^{(0)}|$; if $r_j = 0$ the circle reduces to a point. The usual analysis of frequencies, amplitudes and periodicity or quasiperiodicity can be easily elaborated now.

3.-Finite dimensional projective Hamiltonians. With notations as in the previous section, for each non-zero $y \in E^n$, let $L = [y] = \{\mu y | \mu \in \mathbf{C}\}$ be the complex line through y. The *complex projective space* PE^n is the set of all these lines

$$PE^n = \{[y] | y \in E^n - \{0\}\} \tag{7}$$

This is a compact complex manifold of complex dimension $n-1$ (real dimension $2n-2$) with canonical Hermitian, Riemannian and symplectic structures. Define the *projective flow* $[\exp(-iHt)] : PE^n \to PE^n$ of equation (2) as

$$[\exp(-iHt)][y] = [\exp(-iHt)y] \tag{8}$$

and define the *projective vector field* X_{-iH} on PE^n as the infinitesimal generator of the projective flow. If the tangent bundle of PE^n is explicitly given, then an explicit formula for X_{-iH} as a section of this bundle can be written out; see equation (32). A calculation then proves that X_{-iH} is the Hamiltonian field of the function $\frac{1}{2}e_H$ where

$$e_H(y) = \frac{\langle H(y), y\rangle}{\langle y, y\rangle} \tag{9}$$

is the *energy function* (*Rayleigh quotient*, *normalized mean value*) of the operator H on PE^n. In the geometrical context of [Morse-Cairns] the energy e_H is called *Milnor function*. See also [Milnor]. Summing up, if a linear flow is Hamiltonian then the induced projective flow is likewise Hamiltonian.

The energy e_H has n critical points, namely, the eigenspaces $L_j = [y_j] \in PE^n$, y_j=eigenvector of H. These are critical points of e_H (zeros of the derivative De_H), critical points of X_{-iH} (zeros of the Hamiltonian vector field) and stationary states (fixed points) of the projective flow $[\exp(-iHt)]$. The index of L_j as a critical point of the Morse function e_H equals $2j$; the corresponding critical values are $e_H(L_j) = -\lambda_j$, with $-\lambda_0 =$ absolute minimum and $-\lambda_{n-1} =$ absolute maximum. The critical level set $e_H^{-1}(-\lambda_j)$ is a (singular) real hypersurface (real dimension $2n-3$) in PE^n containing just one critical point, namely, the state L_j.

The projective Hamiltonian flow can be studied in detail using local projective coordinates in PE^n. This is done as follows. Given $[y^{(0)}] = L^{(0)} \in PE^n$ choose an eigenspace L_k not orthogonal to $L^{(0)}$. Such L_k always exists because the eigenspaces span E and $y^{(0)} \neq 0$. Let $L_k^\perp$ be the linear hyperplane which is the orthogonal complement of the line L_k. The trajectory of $L^{(0)}$ is then contained in the set $V = PE^n - PL_k^\perp$ (=set theoretical complement of the projective hyperplane $PL_k^\perp$) which is invariant under the Hamiltonian flow. The set V is called in Algebraic Geometry an *open affine* set. There are *projective coordinate systems* $h : V \to L_k^\perp$ which generalize the classical stereographic projection; see section 13. It follows that when expressed in projective coordinates the field X_{-iH} becomes the linear field

$$-i(H - \lambda_k I) : L_k^\perp \to L_k^\perp \tag{10}$$

which has eigenvalues $-i(\lambda_j - \lambda_k), j = 0, \ldots, k-1, k+1, \ldots, n-1$; see formula (33) in Proposition 3. Therefore, when transformed under projective coordinates, the trajectory of $L^{(0)} \in PE^n$ becomes the trajectory of $h(L^{(0)}) \in L_k^\perp$ under the linear Hamiltonian vector field $-i(H - \lambda_k I)$. Thus, $h(L^{(0)})$ has trajectory contained in a certain torus $T \subseteq L_k^\perp$ and it follows that the trajectory of $L^{(0)}$ is contained $h^{-1}(T) \subseteq V \subseteq PE^n$. The local projective coordinates show that X_{-iH} is a collection of linear Hamiltonians glued together over PE^n.

Note that in local coordinates the trajectories have periodic components with frequencies $|\lambda_j - \lambda_k|/(2\pi)$. But these frequencies depend on the coordinate system. In a different coordinate system, say centered at L_m, the frequencies are $|\lambda_j - \lambda_m|/(2\pi)$. Thus, for projective Hamiltonians the frequencies and periods of trajectories are relative notions that depend on the choice of projective coordinates.

An important property valid in general for Hamiltonian vector fields on symplectic manifolds is that they are energy conservative. That is, taking any initial state in the manifold under consideration, the corresponding trajectory is contained in an energy level hypersurface. As a consequence, small changes in th einitial state of the system, say induced by a small external interaction, can only result in small energy changes. After the small initial change the evolution of the system proceeds according to the Hamiltonian evolution without further energy modifications.

Define a *perturbation* of the state L as a state $\tilde{L}$ close to L. This means that in projective space PE^n the distance from $\tilde{L}$ to L is small. In physical situations, for example an electron bound in an atom, the perturbations can result from small external electromagnetic interactions.

Systems with space of states PE^n, energy e_H and dynamics defined by the flow $[\exp(-iHt)]$ are examples of Hamiltonian systems and verify the above condition about energy conservation. They do not spontaneously radiate and the energy is stable under perturbations: If a given state, say $L \in PE^n$, is perturbed to a new state $\tilde{L}$, there will be an energy change equal to $e_H(\tilde{L}) - e_H(L)$ but as time goes on and the system evolves according to the projective Hamiltonian flow this will be the only energy change. If $\tilde{L}$ is close to L the energy change will be small.

A related property is that PE^n is connected and e_H is continuous and not identically constant ($n \geq 2$). Therefore, the values of e_H fill out an interval of real numbers and under the effect of perturbations the energy differences $e_H(\tilde{L}) - e_H(L)$ can take a continuum of values, not just the discrete set $e_H(L_k) - e_H(L_m) = \lambda_m - \lambda_k$ of critical value differences. In other words, for Hamiltonian systems no energy level is forbiden. This applies to both radiative perturbations $e_H(\tilde{L}) < e_H(L)$ and to absorptions $e_H(\tilde{L}) > e_H(L)$.

4.- Inconsistency of projective Hamiltonian evolution. It is instructive to contrast the mathematical formalism of Hamiltonian systems with the physical properties of the hydrogen atom. Consider, on one hand, dynamical systems having space of states PE^n, dynamics defined by the Hamiltonian vector field X_{-iH} and observable e_H. On the other hand, recall the behaviour of the hydrogen atom. Statements HE.1–HE.4 below make clear that projective Hamiltonian evolution laws are a bad choice to describe atomic phenomena. The statements are typical examples of fundamental contradictions common in Quantism. Note that, although enunciated for finite dimensional systems, the statements apply in the infinite dimensional case as well.

HE.1 *Assume an initial excited stationary state $L^{(0)} = L_k \in PE^n$, $k > 0$ and let a small perturbation result in a nearby state $\tilde{L}$. Then, after certain time interval, $e_H([\exp(-iHt)]\tilde{L})$ assumes a critical value $-\lambda_m$ with $-\lambda_m \leq -\lambda_k$, and remains stationary there. Hence perturbations of stationary states induce an energy change equal to an eigenvalue difference $\lambda_k - \lambda_m$.*

Statement HE.1 is true for atoms and false for the Hamiltonian dynamical system under consideration. It corresponds to discretization of the energy *radiated* by an excited bound electron. For both finite and infinite dimensional Hamiltonian systems the statement is false. This is the first example of fundamental contradictions between quantum evolution equations and the physical behaviour of atoms.

HE.2 *Consider an initial stationary state $L^{(0)} = L_k \in PE^n$, perturb it to a new state $\tilde{L}$ with $e_H(\tilde{L}) = -\tilde{\lambda} > e_H(L_k) = \lambda_k$ and assume that $-\tilde{\lambda}$ is not an eigenvalue of H. Then, under the Hamiltonian evolution, the state $\tilde{L}$ will evolve until a stationary state L_m is reached so that the energy energy of the system becomes an eigenvalue $e_H(L_m) = -\lambda_m$. The total energy change is an eigenvalue difference $\lambda_k - \lambda_m$.*

Property HE.2 corresponds to discretization of the energy *absorbed* by a bound electron. It says that after an absorption process the state reached by the system will evolve radiating energy until a stationary state is reached. The statement is true for atoms but obviously false for Hamiltonian dynamical systems. This is the second example of a fundamental contradiction of Quantum theory.

HE.3 *The state $L = L_0 \in PE^n$ at which e_H attains its absolute minimum is the only stationary state with the property that small perturbations cannot induce radiation.*

This says that the ground state is the *only* state with stable energy, as opposed to excited stationary states, which under small perturbations radiate. This is a contradiction because excited atoms do radiate spontaneously but Hamiltonian systems do not. Again, in Hamiltonian systems *all* states are energy stationary: They evolve without radiating energy. Quantism explains the physical stability of the ground state, as opposed to the unstability of other stationary states, invoking the uncertainty principle.

HE.4 *If a stationary state L_k is perturbed to a nearby state $\tilde{L}$ then, according to certain probabilities that depend on $\tilde{L}$, the trajectory $[\exp(-iht)]\tilde{L}$ will become a new stationary state L_m with $-\lambda_m \leq -\lambda_k$.*

Property HE.4 above refers to 'transition probabilities'. It is true that atoms radiate in an apparently spontaneous manner but, once more, Hamiltonian dynamical systems do not behave that way because they are deterministic and energy conservative. This is the fourth and last contradiction chosen here to illustrate the inconsistency of quantum evolution equations when compared with the physical behaviour of atoms.

It is a fact that HE.1–HE.4 are true for atoms and false for the energy conservative dynamical systems postulated by quantism. They remain false if E^n and $H : E^n \to E^n$ are replaced by an infinite dimensional Hilbert space E and a self-adjoint operator $H : E \to E$. Quantum mechanics has raised property HE.4 to the status of a physical law. This hides the inadequacy of Hamiltonian evolution and creates a contradictory situation. Hamiltonian evolution and transiton probabilities contradict each other. The quantum treatment of HE.4 can be described for finite dimensional systems as follows.

The observable being e_H, each state $L = [y] = [\zeta_0 y_0 + \cdots + \zeta_{n-1} y_{n-1}] \in PE^n$ is considered as a random variable with values in $\{-\lambda_0, \ldots, -\lambda_{n-1}\}$. The probability (notation of formula (5)) that the variable L assumes the value $-\lambda_j$ is $p_j = |\zeta_j|^2/\langle y, y\rangle$. When the system is in state L a 'measurement' of e_H gives the value λ_j with probability p_j.

Since the results of measurements are not eigenvalues but rather *eigenvalue differences*, it is less unnatural to regard L as a random variable with values in the set $\{L_0, \ldots, L_{n-1}\}$, the probability of L_j being p_j. So, state L will perform the famous 'quantum jump' to state L_j with probability p_j and will remain there with probability 1; however, perturbations will result in a state $\check{L}$ near L_j, and $\check{L}$ will jump to another stationary state, and so on. This process continues spontaneously until the ground state L_0 is reached. The probabilities p_j still have to be normalized because L cannot spontaneously jump to L_m if $e_H(L) < -\lambda_m$; otherwise energy is spontaneously created by the system violating the principle of conservation of energy. Therefore, if $e_H(L) < -\lambda_m$ let $\bar{p}_m = 0$, and if $e_H(L) \geq -\lambda_j$ replace p_j with $\bar{p}_j = p_j/\sigma$ where $\sigma = \sum_s p_s$, the sum taken over all s such that $e_H(L) \geq -\lambda_s$. The probabilities of a 'quantum jump' should be given by the $\bar{p}_j$.

But this quantum procedure is unsatisfactory because the Hamiltonian projective vector field is still assumed to govern the system and the probabilistic quantum jumps are in open contradiction with the energy conservative behavior typical of Hamiltonian systems. A better alternative would be the following. Once the critical points L_j are determined, the system formed by the manifold PE^n and observable e_H can be analyzed under the above probabilistic interpretation of states *without* postulating any flow on PE^n. Still the results will resemble HE.1–HE.4. Such hypothetic purely probabilistic theory is a theory without continuous dynamics, that is, no flow is assumed in the space of states. In particular, the Hamiltonian flow is seen to be *useless*. It is not unfair then to say that the probabilistic interpretation of L is an *ad-hoc* argument to force properties HE.1–HE.4 into Hamiltonian systems.

Next, linear and projective vector fields that are the gradients of energy functions will be discussed and their behaviour will be compared with known properties of atoms.

5.-Finite dimensional linear gradients. Assume now that E^n is an n-dimensional Hilbert space, either over the real or the complex number systems. The inner product $\langle , \rangle$ is Euclidean (real) or Hermitian (complex) depending on the scalars. In the complex case the real part of the Hermitian product is an Euclidean product. Hence for any choice of scalars E^n has

an Euclidean inner product and therefore real valued smooth functions with domain an open set in E^n have well defined gradients. The hypothesis on the operator $H : E^n \to E^n$ are as in previous sections: H is self-adjoint with real, negative and simple eigenvalues $-\lambda_0 < \cdots < -\lambda_{n-1} < 0$ and corresponding one-dimensional eigenspaces $L_0, \ldots, L_{n-1}$. This means that H is equivalent in some orthonormal basis to a diagonal matrix with real, negative and unequal elements on the diagonal. The eigenspaces L_j are real or complex lines according to the field of scalars being the real or the complex field.

Consider the linear differential equation defined for $y \in E^n$ by the self-adjoint operator $-H$

$$\frac{dy}{dt} = -H(y) \tag{11}$$

Equivalently, consider the the vector field $-H : E^n \to E^n$. This field is the gradient of the energy function $-\frac{1}{2}e(y)$ where, as before, $e : E^n \to \mathbf{R}$ is given by $e(y) = \langle H(y), y \rangle$. The energy $-\frac{1}{2}e \geq 0$ is an unbounded above Morse function and has a single critical point, namely, a minimum at the origin. The linear flow decomposes as a product of one-dimensional flows

$$\exp(-Ht) = \bigoplus_{j=0}^{n-1} \exp(\lambda_j I_j t) \tag{12}$$

with $I_j : L_j \to L_j$ the identity; therefore the linear flow is *expansive*, $\|U_t(y_1) - U_t(y_2)\| \geq \|y_1 - y_2\|$, and the origin is the only fixed point. All trajectories outside the origin diverge towards infinity and $-\frac{1}{2}e$ increases without bound along them.

Linear expansive flows have a simple geometric interpretation: They are topologically equivalent to the flow $\exp(It)$ generated by the identity operator $I : E^n \to E^n$. Such flows *do not* provide a sound model for microscopic physical systems. Here the dynamics is not better suited to atomic phenomena than the linear or projective Hamiltonian dynamics previously discussed. However it is a fact that the eigenvalue differences of Schrödinger Hamiltonian operator

$$-\nabla^2 + U \tag{13}$$

agree with the energies of photons emitted by the hydrogen atoms and measured in laboratory experiments. Therefore it is natural to look at evolution

equations obtained, one way or another, from operator (13). Schrödinger evolution equation, or equivalently the linear flow $\exp(-i(-\nabla^2+U)t)$, was the dynamics historically selected. But, once more, this Hamiltonian dynamics is not satisfactory.

The non-linear projective flow $[\exp(-(-\nabla^2+U)t)]$, induced on real projective space by the linear flow $\exp(-(-\nabla^2+U)t)$, does not seem to have been considered in microscopic physics as a valid general evolution equation for atomic systems. This projective flow is the alternative to quantum dynamics that will be explored in the present work.

6.-Finite dimensional projective gradients. The assumptions are as in the preceding section. Hence, E^n is an n-dimensional vector space over the real or over the complex numbers and H is self-adjoint with simple negative eigenvalues. For $0 \neq y \in E^n$ let $L = [y] = \{\mu y | \mu \in \mathbf{K}\}$ with $\mathbf{K}$ =field of real or complex numbers. The projective space is therefore

$$PE^n = \{[y] | y \in E^n - \{0\}\}$$

. In the real case (resp. complex case) PE^n is a compact manifold of dimension $n-1$ over $\mathbf{R}$ (resp. dimension 2n-2 over $\mathbf{R}$) and has a canonical Riemannian metric (resp. a canonical Hermitian metric); see formula (40) in section 20. The real part of the Hermitian metric is a Riemannian metric, therefore in both the real and complex cases gradients of real valued differentiable maps with domain PE^n are well defined. The projective flow associated to (11) is defined as

$$[\exp(-Ht)][y] = [\exp(-Ht)y] \tag{14}$$

and the projective field X_{-H} is by definition the infinitesimal generator of this flow.

A computation proves that X_{-H} is the gradient of the function $-\frac{1}{2}e_H$ where the energy map or Rayleigh quotient e_H is given by

$$e_H = \frac{\langle H(y), y\rangle}{\langle y, y\rangle} \tag{15}$$

This is the same expression as (9), but now the domain can be either real or complex projective space. The energy e_H is a Morse function with n critical

points $L_j \in PE^n$, $j = 0, \ldots n-1$. The index of L_j is j (resp. $2j$) in the real case (resp. in the complex case). The critical points are isolated because the eigenvalues are simple.

It follows from rather general arguments of Morse theory that the gradient of the function $-\frac{1}{2}e_H$ has a flow that is hyperbolic at critical points. The stable manifold of L_j has in the real case (resp. in the complex case) dimension $n-j-1$ (resp. dimension $2(n-j-1)$). Thus, the projective flow (14) is as expected for the gradient of a Morse function on a compact manifold. But for a detailed description of projective gradient flows the general apparatus of Morse theory is not necessary. Elementary arguments suffice to prove that projective flows are hyperbolic, to calculate the limits of trajectories and to specify the stable and unstable manifolds.

As was the case with projective Hamiltonians, the projective gradient flows are analyzed by means of local projective coordinates in PE^n. Again, choose an eigenspace L_k not orthogonal to L, and let $h : V \to L_k^\perp$ be the corresponding projective coordinate system. If transformed under these coordinates the projective gradient flow $[\exp(-Ht)]$ becomes the linear flow $\exp(-(H-\lambda_k I)t)$ in $L_k^\perp$. Note that this linear flow is hyperbolic on its domain $L_k^\perp = \bigoplus_{j\neq k} L_j$, with stable and unstable manifolds equal to $\bigoplus_{j>k} L_j$ and $\bigoplus_{j<k} L_j$ respectively. This follows from the signs of the non-zero eigenvalues $-(\lambda_j - \lambda_k) = \lambda_k - \lambda_j$ that correspond with the eigenspaces L_j, $j = 0, \ldots, k-1, k+1, \ldots, n-1$. The projective coordinate transformations show that X_{-H} can be viewed as a collection of linear hyperbolic flows glued together over PE^n. See Proposition 2 on section 14 and Proposition 3 on section 15.

7.-Stable and unstable manifolds. The relation between the linear and projective flows of equation (11) can be described geometrically as follows. Consider formula (5) but let now the coefficients $\zeta_0^{(0)}, \ldots \zeta_{n-1}^{(0)}$ be real or complex scalars. The trajectory of $\exp(-Ht)$ with initial condition $y = y(0) = y^{(0)}$ is

$$y(t) = \sum_{j=0}^{n-1} \exp(\lambda_j t)\zeta_j^{(0)} y_j \tag{16}$$

As $t \to +\infty$ the point $y(t)$ moves along a diverging trajectory and the line

$L^{(t)} = [y(t)]$ pivots around the origin approaching some eigenspace L_k, which for the given initial state $L^{(0)} = [y^{(0)}]$ is determined from (5) by looking for $k =$ minimum of the integers j such that $\zeta_j \neq 0$. The reason for this behaviour is that the one-dimensional components $\exp(\lambda_j t)$ of the linear flow have different rates of growth that depend on λ_j and for $t \to +\infty$ the first nonzero λ_j, namely λ_k, dominates. This implies that in PE^n the stable manifold of the stationary state L_k is the set of the states $L = [y]$ for which $\zeta_0 = \ldots = \zeta_{k-1} = 0$ and $\zeta_k \neq 0$.

In a more concise notation, with k as above, the stable manifold of L_k is $P(L_k + \cdots + L_{n-1}) - PL_k^{\perp}$ (set theoretical difference of projective subspaces); this is an open affine set in $P(L_k + \cdots + L_{n-1})$. By a similar argument the unstable manifold of L_k is $P(L_0 + \cdots + L_k) - PL_k^{\perp}$. In general, given $L = [\zeta_0 y_0 + \cdots + \zeta_{n-1} y_{n-1}] \in PE^n$, if m and k are the maximum and minimum of the integers j with $\zeta_j \neq 0$, then the trajectory $[\exp(-Ht)]L$ comes from L_m and goes to L_k. In other words, L_m and L_k are the *α-limit* and *ω-limit* of the trajectory of L:

$$\begin{array}{rcl} \alpha(L) & = & \lim_{t \to -\infty} [\exp(Ht)]L = L_m \\ \omega(L) & = & \lim_{t \to \infty} [\exp(-Ht)]L = L_k \end{array} \tag{17}$$

Define also the *energy α-limit* and *energy ω-limit* of L as

$$\begin{array}{rcl} e_H^{\alpha}(L) & = & \lim_{t \to -\infty} e_H([\exp(-Ht)]L) = e_H(L_m) = -\lambda_m \\ e_H^{\omega}(L) & = & \lim_{t \to \infty} e_H([\exp(-Ht)]L) = e_H(L_k) = -\lambda_k \end{array} \tag{18}$$

8.-Energy changes. If the trajectory of $[\exp(-Ht)]L$ comes from its α-limit L_m and goes to its ω-limit L_k then the *energy change along the trajectory* is defined as $e_H(L_k) - e_H(L_m) = -\lambda_k - (-\lambda_m) = \lambda_m - \lambda_k$, an eigenvalue difference.

The Principle of Conservation of Energy implies that, given the energy of an initial state, any additional energy that the system aquires has to be absorbed from outside the system and any energy that the system losses is radiated outside the system. If the energy change of the system is negative then the outside gains energy. If the energy change in the system is positive then the outside losses energy. Energy changes of the system that are equal

to an eigenvalue difference corresponds to an eigenvalue difference of energy radiated or absorbed by the system, with sign conventions distinguishing the two cases.

Consider now an initial stationary state L_p and perturb it to a nearby state $\tilde{L}_p$. Assume that the energy difference is small: $|e_H(\tilde{L}_p) - e_H(L_p)| < \epsilon$. If the ω-limit of $\tilde{L}_p$ is L_k then, for $\tilde{L}_p$ not too far from L_p, one has $k \leq p$. Therefore the system will evolve towards L_k with energy change $e_H(L_k) - e_H(\tilde{L})_p$ and this is equal, up to ϵ, to the eigenvalue difference $\lambda_k \lambda_m$. This means that the total amount of energy the system radiates when perturbed away from a stationary state is almost exactly an eigenvalue difference. Physically, the small difference ϵ is usually attributed to the 'finite bandwidth ' of spectral lines or to 'background noise'. The discussion illustrates how energy discretization is a natural consequence of projective gradient dynamics. Starting from a perturbed stationary state the system dissipates energy in the most efficient way (down the energy gradient) and the state evolves continuously until a new stationary state, a state locked in itself, is attained. An eigenvalue difference of energy is radiated. If the new stationary state is not the ground state then it is unstable: Small perturbations destabilize the state and again continuous evolution way down the energy gradient will radiate a quantity of energy equal to an eigenvalue difference. The process repeats until the ground state is reached.

This contrasts with the Hamiltonian evolution equations of quantism that cannot predict energy discretization and in order to mimic atomic behaviour require especially tailored quantization postulates. Still worse, the postulates do not complement but rather *contradict* the Hamiltonian evolution equations. If evolution occurs according to projective Hamiltonian equations then transitions between stationary states, and probabilistic quantum jumps in particular, cannot occur. Allegedly non-deterministic and discontinuous phenomena, considered by quantism a paradigm of Nature, are in reality imaginative contradictions of a mistaken theory.

In projective gradient systems, besides the total energy change along a trajectory, energy can also be considered as a function of time. These considerations are impossible in the energy conservative Hamiltonian systems of Quantism. So, let $L = L^{(0)} \in PE^n$ be an arbitrary initial state (at time 0)

so that at time $t > 0$ the state of the system becomes $L^{(t)} = [\exp(-Ht)]L^{(0)}$. The energy *radiated* up to time t by this trajectory is by definition

$$G(t) = -(\, e_H(L^{(t)}) - e_H^{\alpha} L^{(0)})\,) \tag{19}$$

and the *intensity* or *output rate* of the energy being radiated at instant t equals the derivative $g(t) = G'(t)$. A computation gives

$$g(t) = 2(e_{H^2} - e_H^2)(L^{(t)}) \tag{20}$$

In the wave theoretical (infinite dimensional) case the physical assumption can be made that the energy radiated by an atom propagates in ordinary space with constant direction and speed c. The energy distribution along the axis has, at point $x = ct$, density $(1/c)g(t/c)$. This provides both a proof and an interpretation of Einstein formula

$$\Delta\lambda = \frac{hc}{\Lambda} \tag{21}$$

relating the energy $\Delta\lambda$ and the wavelenght Λ of a photon. For details see section 25 below and Chapters 4 and 5 of [1]. The functions $g(t)$ and $(1/c)g(t/c)$ are *energy packets*.

The following additional terminology is illustrative. It translates physical phenomena into dynamical properties of flows. The term 'almost' will be abbreviated 'a-'. An *a-stationary* is a state $\tilde{L}_j$ close to a stationary state L_j. An *a-transition*, denoted $L_m \to L_k$, is a trajectory $[\exp(-Ht)L]$ with $t_0 \le t \le t_1$ and such that $\tilde{L}_m = [\exp(-Ht_0)]L$ is an a-stationary state close to L_m and $\tilde{L}_k = [\exp(-Ht_1)]L$ is an a-stationary state close to L_k; t_0 and t_1 are the initial and final times of the a-transition.

If an a-transition is between adjacent a-stationary states, that is, if $m = k+1$, then the energy output rate $g(t)$ has a single maximum and no local minima in the open interval (t_0, t_1). If $m = k + 2$ and if the trajectory from $\tilde{L}_{k+2}$ to $\tilde{L}_k$ is away from the intermediate stationary state L_{k+1} then again there is a single maximum; but if the trajectory passes near L_{k+1} then $g(t)$ decreases there to grow later again and therefore will have two local maxima with a local minimum in between. In case $m = k + d$ with $d > 1$ there are a

number of possibilities for $g(t)$, depending on the proximity of the trajectory $[\exp(-Ht)]\tilde{L}_m$ to the stationary states L_j, $k < j < m$. These possibilities range from a single maximum to d local maxima and $d-1$ local minima in the open interval (t_0, t_1). An a-transition $L_m \to L_k$ can look, from the viewpoint of radiated energy, as several successive a-transitions $L_m \to L_{m-1} \to \cdots \to L_k$.

9.- Consistency of projective gradient evolution. In statements HE.1–HE.4 physical properties of microscopic systems were found in contradiction with Hamiltonian evolution. Four similar statements GE.1–GE.4 will be now formulated and the discussion will show that the physical properties in question are consistent with projective gradient evolution. The relevant mathematical properties of projective gradient systems have been explained in sections 6–8.

GE.1 *Consider an initial excited stationary state $L = L_m$ with $m > 0$. If a small perturbation produces a nearby state $\tilde{L}_m$ then the system will evolve towards the stationary state $L_k = \omega$-limit of $\tilde{L}_m$, $k \leq m$. The total energy radiated is, up to a small term, an eigenvalue difference $\lambda_k - \lambda_m$.*

This says that the radiated energy is discretized, a typical behaviour of electrons bound in atoms. For projective gradient evolution the hyperbolicity of L_m is the qualitative reason for spontaneous radiation: Either the initial state is very close to a stationary one, but is not exactly stationary, or small perturbations destabilize the stationary state and result in energy radiation.

Consider again an initial stationary state L_m and let the system absorb energy. It can be assumed that this energy has been radiated from some source under the form of an energy packets as in formula (20), the total energy of the packet being λ. The absorption process forces the system to evolve along certain curve in PE (a curve that is *not* a trajectory of the projective gradient field X_{-H}). The interaction with the incoming energy packet produces this behaviour. Once all the incoming energy is absorbed a certain state L is reached by the system. Conservation of energy requires that $e_H(L) = \lambda_m + \lambda$. If λ is not an eigenvalue difference then $e_H(L)$ cannot be an eigenvalue of H nor can L be stationary. The system will immediately radiate energy back. Hence, the energy absorbed will not result in a stationary state unless

λ =eigenvalue difference. Therefore systems that obey a projective gradient evolution law absorb energy in discrete packets.

Note that there exist states that are non-stationary but have energy equal to an eigenvalue, $-\lambda_m$. Therefore an incoming amount of energy equal to an eigenvalue difference does not necessarily result in the system reaching a stationary state. The condition is necessary but not sufficient.

Once a non-stationary state L has been reached at the completion of the absorption process there are several possibilities for the the behaviour of the system, depending on the first a-stationary state $\tilde{L}_p$ reached by the system, begining in L and evolving way down the energy gradient.

a) If $p = m$ then the system returns near the initial stationary state L_m radiating an amount of energy (almost) equal to λ. This resembles *elastic* or *Rayleigh* scattering.

b) If $p > m$ then the system will radiate energy by an amount $\lambda - (\lambda_p - \lambda_m) < \lambda$. This is similar to Raman scattering with Stokes line.

c) If $p < m$ then the system will radiate energy by an amount $\lambda - (\lambda_p - \lambda_m) > \lambda$. The situation is like Raman scattering with anti-Stokes line.

These considerations lead to the following

GE.2 *Let L_m be an initial stationary state and let the system absorb an amount of energy equal to λ. If λ is not an eigenvalue difference, $\lambda \neq \lambda_m - \lambda_k$, no stationary state is reached and energy is radiated back. A necessary condition for a stationary state to be reached is that the amount of incoming energy be an eigenvalue difference, $\lambda = \lambda_m - \lambda_k$.*

Statement GE.2 is the discretization of absorbed energy. Compare with statement HE.2.

GE.3 *The ground state L_0 is the only stable stationary state.*

Statement GE.3 is true because the flow $[\exp(-Ht)]$ is generated by the

gradient of the energy function $-\frac{1}{2}e_H$ and this function has a unique stable critical point L_0, an absolute minimum. L_0 is the ground state and does not radiate because it is the bottom of an energy well. The stability of the ground state is a most natural consequence of gradient evolution . No uncertainty principle is required. Compare with the corresponding Hamiltonian case HE.3.

GE.4 *Given a stationary state L_m and $k < m$, any neighborhood of L_m contains perturbations $\tilde{L}_m$ that make a-transitions to L_k.*

Therefore for $k < m$ and with sensitive dependency on the perturbation, there are a-transitions $L_m \to L_k$. Statement GE.4 is a particular case of properties discussed in section 8. More specifically, let $L_m = [y_m]$, $L_k = [y_k]$ and $\tilde{L}_m = [ay_m + by_k]$. Given any neighborhood of L_m in PE^n, if b/a is close enough to 0 the state $\tilde{L}_m$ belongs to the neighborhood and the trajectory $[\exp(-Ht)]\tilde{L}_m$ is an a-transition as required. Note that this is in fact an exact transition since the ω-limit of the trajectory is $\omega(\tilde{L}_M) = L_k$.

If sensitive dependency on initial conditions is taken as the definition of chaos then statement GE.4 says that projective gradient flows are chaotic. This determistic and continuous but chaotic behaviour at critical points has been explained by quantism as an intrinsically random and discontinuous physical phenomenon. Compare with statement HE.4.

Gradient projective vector fields can be used instead of projective Hamiltonians and the resulting dynamics reflects more accurately the known behavior of atoms. It is no longer necessary to have a probabilistic interpretation for states and the flow plays its natural role, namely, to tell how states evolve. All the arguments given so far generalize to the infinite dimensional case $H = -\nabla^2 + U$ =Hamiltonian of the hydrogen atom.

10.- Conclusions from finite dimensional cases. The various dynamical systems considered so far can now be compared. The linear flow $\exp(-iHt)$ and the induced projective flow $[\exp(-iHt)]$ are defined in symplectic manifolds and are Hamiltonian. There is one critical point in the linear case and n in the projective case. In general, trajectories of Hamiltonian flows are contained in energy hypersurfaces (surfaces of constant energy). The projective

Hamiltonian has a more sophisticated dynamics than the linear Hamiltonian. Nevertheless, these Hamiltonian flows contradict the most basic properties of atoms.

The linear flow $\exp(-Ht)$ and the induced projective flow $[\exp(-Ht)]$ are gradients of Morse functions with one and n hyperbolic critical points respectively. The linear gradient has unbounded divergent trajectories and the projective gradient has trajectories that go from one critical point to another. All these trajectories dissipate energy. Again, the projective gradient flow has a more elaborated geometry than the linear gradient flow. Projective gradient flows predict, much better than projective Hamiltonian flows, fundamental properties of atomic behaviour.

The projective flows $[\exp(-iHt)]$ and $[\exp(-Ht)]$ have the same critical points on PE^n. However, as shown in the previous sections, these two flows have very different dynamical structures that reflect the deep difference between standard quantum mechanics and Realism.

11.-Energy output rate. Let X be a vector field, $\kappa > 0$. Then X and κX have the same trajectories except for reparametrization. Hence, if in the previous discussion e_H is left the same and the projective flow $[\exp(-Ht)]$ is replaced by $[\exp(-\kappa Ht)]$ then equation (20) becomes

$$g(t) = 2\kappa(e_{H^2} - e_H^2)(L^{(t)}) \tag{22}$$

In physical situations one has H =Hamiltonian operator and κ is a constant with dimensions 1/[Action]. If the value of κ is postulated as

$$\kappa = \frac{2}{h} \tag{23}$$

with h=Planck constant then Einstein formula (21) can be proved and interpreted. In the finite dimensional case the value $\kappa = 1$ has been assumed in order to simplify the discussion.

Part II. Infinite Dimensional Projective systems

Projective vector fields, projective flows and energy maps defined in projective space will be now discussed in the infinite case dimensional case. These projective systems will be constructed starting from a single self-adjoint operator on Hilbert space.

In linear operator theory basic versions of the Spectral Theorem apply to operators $H : E \to E$ in the following cases:

1.- E is complex and H is self-adjoint.
2.- E is complex and H is normal.
3.- E is real and H is self-adjoint.
4.- E is real and H is normal.

The version of the Projective Spectral Theorem presented here, Theorem 1 below, applies as follows: In cases 1 and 2, to the flow $[\exp(-Ht)]$ on complex projective space. In case 3 to the flow $[\exp(-Ht)]$ on real projective space.

12.-Terminology and notation. A general source for the mathematical theories of Global Analysis is [Abraham]. Let E, F be Hilbert spaces over the real numbers or over the complex numbers. Denote by $\mathrm{Hom}(E, F)$ the (Banach) space of continuous linear maps from E to F and by $I = I_E : E \to E$ the identity map. The inner product, whether real or Hermitian, will be denoted $\langle\ ,\ \rangle$. Basic examples of particular interest here are $\mathrm{L}^2_{\mathbf{C}}(\mathbf{R}^3)$ and $\mathrm{L}^2_{\mathbf{R}}(\mathbf{R}^3)$, the spaces of real and of complex valued square integrable functions with domain $\mathbf{R}^3$. Vector subspaces of E are always assumed to be closed and linear maps, operators or linear forms are continuous. If F is a subspace of E let $F^\perp$ be the orthogonal complement of F in E and let $Q_F : E \to E$ be the orthogonal projection over F with kernel $F^\perp$. For sets S_1, S_2, $pr_j : S_1 \times S_2 \to S_j$, $j = 1, 2$, are the canonical projections $pr_j(s_1, s_2) = s_j$.

A *smooth map* is a function with domain an open set in some Banach space, with values also in a Banach space and such that derivatives of all orders exist at all point of the domain. For U open in E and $f : U \to F$ a smooth map, the derivative of f at $u \in U$ is a linear map $Df(u) : E \to F$. A manifold M means a smooth manifold modelled in some Banach space. For $x \in M$, T_xM is the tangent space at x. Let $\tau M = (TM, q, M)$ be the tangent vector bundle of M. Here $TM = \bigcup_{x \in M} T_xM$ is the space of all tangent vectors

and $q : TM \to M$ is the projection. Maps $f : M \to M'$ between manifolds are smooth if whenever expressed in local coordinates the result is a smooth map between Banach spaces; the derivative of such f at $x \in M$ is then a linear map $Df(x) : T_xM \to T_{f(x)}M'$. The *bundle derivative* of f is the vector bundle map $Tf : TM \to TM'$ defined by the condition $Tf|T_xM = Df(x)$.

Vector bundles are always smooth and locally trivial, and will be denoted $\xi = (B, q, Y)$, with B =total space, Y =base space, q =projection and $F_y = q^{-1}(y)$ =fibre over $y \in Y$. So, B and Y are smooth manifolds and F_y is a vector space. It will be assumed that local trivializations of vector bundles can be chosen with typical fibre a Banach space. The total space and projection of the vector bundle ξ will also be denoted B_ξ and q_ξ. A *bundle operator* or *bundle endomorphism* is a map $f : B \to B$, sometimes denoted $f : \xi \to \xi$, such that $q \circ f = q$ and which is linear on fibres. The identity bundle operator $I_\xi : \xi \to \xi$ has $f = I_B : B \to B$, the identity map.

A smooth map $U : M \times \mathbf{R} \to M$ is a *flow* or *one-parameter group* on M if the collection $U_t : M \to M$, $t \in \mathbf{R}$, satisfies the conditions $U_0 = I_M$ and $U_{t_1} \circ U_{t_2} = U_{t_1+t_2}$. Here U_t is defined by the formula $U_t(x) = U(t, x)$. A flow in a Hilbert or Banach space E is a *linear flow* if each $U_t : E \to E$ is linear.

Recall that scalars are real or complex, depending on the Hilbert space under consideration being a real or a complex vector space. Let μ be a scalar. For a Hilbert space E with identity operator $I_E : E \to E$ define the *scalar linear flow* as the flow $\exp(\mu It) = (e^{\mu t})I$. This flow is well defined even if E is just a vector space without norm.

Let $\xi = (B, q, Y)$ be a vector bundle with fibres F_y, $y \in Y$. A *bundle flow* is a flow U on B such that $U_t : \xi \to \xi$ is a bundle operator. Define the *scalar bundle flow* as the unique flow $\exp \mu I_\xi t : \xi \to \xi$ which on each fibre F_y equals the scalar linear flow $\exp(\mu I_y t)$. Here I_y=identity operator on F_y.

13.-Projective spaces and charts. The *projective space* associated to E is

$$PE = \{L \subset E | \dim L = 1\}$$

Equivalently, for $\psi \in E - \{0\}$ let $[\psi]$ denote the line through ψ, then

$$PE = \{[\psi] | \psi \in E - \{0\}\}$$

Projective spaces are smooth manifolds modelled in hyperplanes of E. Each $\psi_0 \in E - \{0\}$ defines a chart. To see this, let $L_0 = [\psi_0]$ and consider the *open affine set* $V_0 = PE - PL_0^\perp$ (set theoretical difference). Then the chart $h_0 : V_0 \to L_0^\perp$ is defined for $L \in V_0$ by the expression

$$h_0(L) + \psi_0 = (L_0^\perp + \psi_0) \cap L \tag{24}$$

or equivalently, for $\langle \psi, \psi_0 \rangle \neq 0$,

$$h_0([\psi]) = \frac{\langle \psi_0, \psi_0 \rangle \psi \ - \ \langle \psi, \psi_0 \rangle \psi_0}{\langle \psi, \psi_0 \rangle} \tag{25}$$

The chart h_0 is a *projective coordinate system.* It can be seen that if a pair of open affine sets overlap then the corresponding coordinate change is smooth. If F is a (closed) subspace of E then PF is a submanifold of PE. The tangent bundle of PE is denoted $\tau PE = (TPE, q, PE)$.

14.-Projective flows. Let $A : E \to E$ be a bounded linear operator with (bounded set of) eigenvalues $\lambda_0, \lambda_1, \ldots$ forming a discrete set in $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. So, the eigenvalues are real if E is a real Hilbert space, and complex if E is complex. Let the corresponding eigenspaces be $F_0, F_1, \ldots$ with identity operators to be denoted $I_j : F_j \to F_j$. Throughout this paper the assumption will be made that the eigenspaces E_j are mutually orthogonal and span E. In other words, A is diagonalizable over some orthonormal basis. The basic example to keep in mind is $E = L^2_{\mathbf{K}}(\mathbf{R}^3)$ =space of square integrable $\mathbf{K}$-valued functions defined on $\mathbf{R}^3$, $A = H$ =Schrödinger hamiltonian operator for a bound system, say, for the hydrogen atom. The above hypothesis on A the linear flow $\exp(At)$ is defined for all t and splits as an infinite product of one-dimensional flows

$$\exp(At) = \bigoplus_{j=0}^{\infty} \exp(\lambda_j I_j t) \tag{26}$$

Note that the subspace generated by any subcollection of the the F_j's is invariant under A and therefore under $\exp(At)$. In particular orthogonal complements of eigenspaces, $F_k^\perp = \bigoplus_{j \neq k} F_j$, are invariant.

The operator A can be considered as a linear vector field on E and the solution curve of this vector field for a given initial condition $\psi \in E$ is $t \to \exp(At)\psi$. Note that for $\kappa \neq 0$, A and κA have the same eigenvectors while the eigenvalues of κA are $\kappa\lambda_j$.

Since $\exp(At)$ is a one-parameter linear group on E it induces a flow $[\exp(At)]$ on PE, the *projective flow* of A, defined for $\psi \in PE$ by the formula

$$[\exp(At)][\psi] = [\exp(At)\psi] \tag{27}$$

Let $L = [\psi] \in PE$ be a stationary state (fixed point) of the projective flow, that is, let $[\exp(At)\psi] = [\psi]$ hold for all $t \in \mathbf{R}$. Then there is a smooth scalar valued function $\lambda(t)$ with $\lambda(0) = 1$ such that $\exp(At)\psi = \lambda(t)\psi$. Taking derivatives on both sides it follows that $A(\exp(At)\psi) = \lambda'(t)\psi$, hence $A(\lambda(t)\psi) = \lambda'(t)\psi$. But then $(\lambda'(t)/\lambda(t))\psi = A(\psi)$ is independent of t. Therefore $\lambda(t) = \exp(\lambda(0)t)$ and $A(\psi) = \lambda'(0)\psi$. This proves the following:

Proposition 1. *$L = [\psi] \in PE$ is a stationary state of the projective flow $[\exp(At)]$ if and only if L is an eigenspace of A. Equivalently, if and only if ψ is an eigenvector of A.*

Therefore in order to explicitly determine the fixed points of $[\exp(At)]$ it is necessary to explicitly solve the eigenfunction problem $A(\psi) = \lambda\psi$. Note that, by Proposition 1, the set of stationary states of $[\exp(At)]$ is the disjoint union of the projective subspaces $\bigcup_{j=0}^{\infty} PF_j$. Therefore $L_k \in PF_k$ is an isolated stationary state if and only if λ_k is a simple eigenvalue, if and only if $PF_k = \{L_k\}$.

Remark. Proposition 1 provides a proof of the Fundamental Theorem of Algebra. Knowledge of the Euler characteristic of complex projective spaces and of Hopf Theorem on the existence of zeros of vector fields over manifolds with non-zero Euler characteristic are also required. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial, $n > 0$. The companion matrix (or Frobenius matrix) of the polynomial $p(z)$ has characteristic polynomial equal, up to sign, to $p(z)$ itself. Let A be the linear endomorphism of $\mathbf{C}^n$ defined by the matrix and let X_A be the associated projective vector field on $P\mathbf{C}^n$. Since the Euler characteristic of $P\mathbf{C}^n$ is $n \neq 0$ it follows that

X_A has a zero. By Proposition 1, A has an eigenvector and therefore $p(z)$ has a root.

Let $[\psi_k] = L_k \in PF_k$ be a stationary state of $[\exp(At)]$, $V_k = PE - PL_k^\perp$ the corresponding open affine set. The projective coordinate system $h_k : V_k \to L_k^\perp$ defined by ψ_k and centered at L_k transforms the projective flow $[\exp(At)]$ into a certain flow $U_t : L_k^\perp \to L_k^\perp$. For $\eta \in L_k^\perp$ these are related by the formula

$$U_t(\eta) = h_k([\exp(At)]h_k^{-1}(\eta)) \tag{28}$$

From equation (25) it follows that

$$h_k^{-1}(\eta) = [\eta + \psi_k] \tag{29}$$

and a calculation then gives

$$U_t(\eta) = \exp(At\lambda_k It)\eta \tag{30}$$

This proves

Proposition 2. *If transformed under the projective coordinate system centered at the stationary state $L_k \in PF_k$, the projective flow $[\exp(At)] : PE \to PE$ becomes the linear flow $\exp((A - \lambda_k I)t) : L_k^\perp \to L_k^\perp$.*

Since the eigenspaces of A span E the domains of projective coordinate systems centered at fixed points cover PE, therefore Proposition 2 means that the projective flow can be considered as a collection of linear flows glued together over PE.

15.-Projective fields. By definition the *projective field* X_A induced by A is the infinitesimal generator of the projective flow $[\exp(At)]$. Thus, X_A is a section of the tangent bundle τPE. For the tangent space to PE at L there is a canonical isomorphism

$$T_L PE \cong \mathrm{Hom}(L, L^\perp) \tag{31}$$

which gives rise to an explicit formula for X_A, namely

$$X_A(L)\psi = Q_{L^\perp}(A(\psi)) \tag{32}$$

Here $X_A(L) \in \mathrm{Hom}(L, L^\perp)$, $\psi \in L$ and $Q_{L^\perp}$ is the orthogonal projection with image $L^\perp$. Proof of the isomorphism is given in section 29.

Formula (32) implies that X_A depends linearly on A, that is, $X_{A+A'} = X_A + X_{A'}$ and $X_{\lambda A} = \lambda X_A$. The stationary states of X_A are the fixed points of $[\exp(At)]$, given in Proposition 1. The following is a consequence of Proposition 2:

Proposition 3. *If expressed in the projective coordinate system centered at the stationary state $[\psi_k] = L_k \in PF_k$ the projective field X_A becomes the linear vector field*

$$A - \lambda_k I : L_k^\perp \to L_k^\perp \ . \tag{33}$$

Therefore the projective field can be considered as a collection of linear vector fields glued together over PE.

16.-Projective flows and normal bundles. This section describes the bundles and flows that appear in Theorem 1, section 18.

The following has been established so far for the projective flow $[\exp(At)]$. The stationary states are given by eigenvectors of A (Proposition 1), near the stationary state L_k the flow $[\exp(At)]$ is equivalent to the linear flow $\exp((A - \lambda_k I)t)$ (Proposition 2) and, again near the stationary state, the infinitesimal generator X_A is equivalent to the linear vector field $A - \lambda_k I$ (Proposition 3).

Consider an eigenvalue λ_k with eigenspace F_k. All $[\psi_k] \in PF_k$ are critical points, hence the flow is constant on PF_k. It turns out that the invariant open set $PE - PF_k^\perp$ provides a convenient neighborhood of PF_k in PE to analize the behaviour of $[\exp(At)]$ around the fixed point set PF_k. It will be proved in the next section that this open set is in a natural way the total space of a vector bundle over PF_k. Denote this bundle by $\nu(F_k, E)$. The notation refers to the fact that the bundle can be identified with the normal bundle to the submanifold PF_k in PE.

For the special case $\dim F_k = 1$ the projective subspace PF_k reduces to a

point and $V = PE - PF_k^{\perp}$, an invariant open affine set already defined in section 13 as domain of a projective chart, is the total space of a vector bundle over the single point $L_k \in E$. For the general case, that is, for F_k of arbitrary dimension, $PE - PF_k^{\perp}$ can be considered as a generalization of the open affine sets that provide domains for the projective coordinate systems.

In the next section it will be seen that each fiber of the bundle $\nu(F_k, E)$ is invariant under $[\exp(At)]$ and, furthermore, that the flow is linear on the fibres. Therefore the projective flow restricts in $PE - PF_k^{\perp}$ to a bundle flow. The Hilbert space splitting $F_k^{\perp} = \bigoplus_{j \neq k} F_j$ induces a vector bundle splitting $\nu(F_k, E) = \bigoplus_{j \neq k} \nu(F_k, F_k \oplus F_j)$ and on each of the components of this splitting the flow is a scalar bundle flow $\exp((\lambda_j - \lambda_k)It)$. This splitting of $[\exp At]$ as sum of scalar bundle flows is the Projective Spectral Theorem of section 18. The projective flow induced by a diagonalizable operator A is explicitly described in terms of objects constructed from spectral data of A, namely, in terms of eigenvalue differences and vector bundles constructed from the eigenspaces. As a consequence the projective flow induced by a self-adjoint operator is hyperbolic. See section 27.

17.-Bundle structures. Let F be a proper closed subspace of E so that $E = F \oplus F^{\perp}$ and consider the triple $\nu(F, E) = (PE - PF^{\perp}, q, PF)$. Here $q : PE - PF^{\perp} \to PF$ maps the line $L' \in PE - PF^{\perp}$ to the line $Q_F(L') \in PF$. It will be proved that $\nu(F, E)$ is a vector bundle with typical fibre $F^{\perp}$. This will be done in two steps. First, vector space structures will be defined on the fibres. Second, local trivializations will be constructed.

For the first step, note that the fiber of q over $L \in PF$ is $q^{-1}(L) = P(L \oplus F^{\perp}) - PF^{\perp}$ and that it can be put into one-to-one correspondence with the vector space $\mathrm{Hom}(L, F^{\perp})$ by sending a point L' in the fiber to the unique linear map with graph L'. This correspondence is natural and defines the vector space structure on $q^{-1}(L)$. Since L is one-dimensional $\mathrm{Hom}(L, F^{\perp})$ is isomorphic to $F^{\perp}$ and therefore $F^{\perp}$ can be taken as the typical fiber. Note that the zero of the vector space $q^{-1}(L)$ is L itself. The argument implies a natural disjoint union decomposition of $PE - PF^{\perp}$ as

$$PE - PF^{\perp} \cong \bigcup_{L \in PF} \mathrm{Hom}(L, F^{\perp}) \tag{34}$$

For the second step, given $L \in PF$ let $h : V = PE - PL^\perp \to L^\perp$ be a projective coordinate system centered at L. Note that $q^{-1}(PF - PL^\perp) = PEPL^\perp$ and $L^\perp = (L^\perp \cap F) \oplus F^\perp \simeq (L^\perp \cap F) \times F^\perp$, the isomorphism being canonical. The following diagram then commutes and provides a trivialization of $\nu(F, E)$ over the open subset $PF \cap V$ of the base space PF:

$$\begin{array}{ccc} PE - PF^\perp & \longrightarrow & (L^\perp \cap F) \times F^\perp \\ q \downarrow & & \downarrow pr_1 \\ PF \cap V & \longrightarrow & L^\perp \cap F \end{array} \tag{35}$$

The horizontal arrows are restrictions of the projective local coordinate h. A routine calculation shows that for these local trivializations the transition functions are smooth and linear on fibers. This completes the construction of the bundle $\nu(F, E)$. It will be proved later that this is in fact the normal bundle of PF in PE.

18.-Projective Spectral Theorem. One possible formulation of the elementary classical linear Spectral Theorem is the formula

$$\exp(At) = \bigoplus_{j=0}^{\infty} \exp(\lambda_t) \tag{36}$$

This expresses the linear flow as a superposition of scalar flows. For the projective flow let λ_k be an eigenvalue of A with eigenspace F_k and consider the bundle $\nu(F_k, E)$. Let $F = F_k$ in diagram (35) and recall that the horizontal arrows are restrictions of h. Proposition 2 implies that $[\exp(At)]$ corresponds under h with the linear flow $\exp((A - \lambda_k I)t)$. Because $F_k^\perp$ is A-invariant, it follows that each fiber of $\nu(F_k, E)$ is invariant under $[\exp(At)]$ and that the projective flow is linear in these fibers.

The Hilbert space splitting $F_k^\perp = \bigoplus_{j \neq k} F_j$ induces a splitting $\mathrm{Hom}(L, F_k^\perp) = \bigoplus_{j \neq k} \mathrm{Hom}(L, F_j)$ and this gives a bundle splitting

$$\nu(F_k, E) \simeq \bigoplus_{j \neq k} \nu(F_k, F_k \oplus F_j) \; . \tag{37}$$

Apply now Proposition 2 to $A|(F_k \oplus F_j)$ to conclude that on each fiber of $\nu(F_k, F_k \oplus F_j)$ the projective flow $[\exp(At)]$ equals the scalar flow $\exp((\lambda_j - \lambda_k)It)$.

Let $I_{k,j}$ denote the identity bundle operator of $\nu(F_k, F_k \oplus F_j)$. The previous arguments prove

Theorem 1. Projective Spectral Theorem: *On the total space* $PE - PF_k^\perp$ *of the bundle* $\nu(F_k, E)$

$$[\exp(At)] = \bigoplus_{j \neq k} \exp((\lambda_j - \lambda_k) I_{k,j} t) \tag{38}$$

In this theorem the projective flow is expressed as a superposition of scalar bundle flows. If E is finite dimensional and $\dim F_k > 1$ the bundle $\nu(F_k, E)$ is non-trivial, that is, not a product bundle and can be called a 'generalized Möbius strip'. Similarly, in the infinite dimensional case there seems to be no natural way to present $\nu(F_k, E)$ as a product bundle. The open dense set $PE - PF_k^\perp$ is not contained in the domain of any natural coordinate system. Therefore the formula in Theorem 1 can be called *semi-global* formula.

If the operator A is skew-adjoint, from formula (38) it is possible to determine relative periods of trajectories, periodicity or cuasiperiodicity and invariant tori.

If A is a self-adjoint operator, from the semiglobal formula it can be concluded that the projective flow is hyperbolic. This means that each connected component of the critical point set is a smooth manifold and that over each fibre of the normal bundle the flow is hyperbolic. For more on hyperbolicity of the stationary manifolds PF_j see section 27. The self-adjoint PST plays an important role in Realism.

Note that if A is replaced by κA, $\kappa > 0$ then $[\exp(At)]$ and X_A become replaced by $[\exp(\kappa At)]$ and κX_A respectively, and the eigenvalue differences become $\kappa(\lambda_j - \lambda_k)$. Therefore the trajectories of the projective flow become reparametrized but otherwise remain the same.

19.-Energy. Assume in sections 19–30 that $H : E \to E$ satisfies the assumptions made in section 14 and that furthermore the eigenvalues are real and negative, $-\lambda_0 < -\lambda_1 < -\lambda_2 < \cdots < 0$, with corresponding eigenspaces $F_0, F_1, F_2, \ldots$. Equivalently, assume that H is self-adjoint and negative on

the real or complex Hilbert space E.

The energy of H is, by definition, the smooth function $e_H : PE \to \mathbf{R}$ given by

$$e_H[\psi] = \frac{\langle H\psi, \psi\rangle}{\langle \psi, \psi\rangle} \tag{39}$$

an expression already considered in Part I of this paper and known as the *Rayleigh quotient* of H.

Starting in the present section and up to section 28 the discussion will center on the dynamics of the flow $[\exp(-\kappa Ht)]$ and its relation with the energy observable e_H. Particularly relevant will be to understand the behaviour of the observable e_H along the trajectories and near the fixed points of the projective flow. The universal constant $\kappa = \frac{2}{h} > 0$, h =Planck constant, gives the rate at which the system outputs energy. From a physical perspective κ plays a most fundamental role.

20.-Projective fields and gradients. In section 14 it was stated that the tangent space to PE at $L \in PE$ is canonically isomorphic to $\mathrm{Hom}(L, L^\perp)$, a fact to be proved in section 29. The inner product on the Hilbert space E induces an inner product on $\mathrm{Hom}(L, L^\perp)$ as follows: Choose a unit vector $u \in L$ and for $\theta, \theta' \in \mathrm{Hom}(L, L^\perp)$ define

$$\langle \theta, \theta'\rangle_{\mathrm{Hom}(L,L^\perp)} = \langle \theta(u), \theta'(u)\rangle_E \tag{40}$$

The value of (40) does not depend on the choice of u. If the scalars are real this formula defines a Riemannian metric on PE. If the scalars are complex it defines an Hermitian metric which is Kähler and has real part equal to a Riemannian metric. Furthermore the Kähler structure defines a symplectic structure. Hence, for any choice of scalars smooth functions on PE have well-defined gradients and, for complex scalars, smooth functions have well-defined Hamiltonian fields.

Lemma 1 (Gradient Lemma): The projective field associated to a self adjoint operator equals the opposite of half the energy gradient:

$$X_{-\kappa H} = (-\frac{\kappa}{2})\nabla e_H$$

Proof: The proof is in section 30.

Recall that, by definition, $X_{-\kappa H}$ is the infinitesimal generator of the projective flow $[\exp(-\kappa Ht)]$, while e_H has been defined as the Rayleigh quotient of H. The Gradient Lemma implies that the critical points of e_H are the stationary states of $[\exp(-\kappa Ht)]$ and by Proposition 1 these are the one-dimensional eigenspaces of H, that is, the elements of the set $\bigcup_{j\geq 0} PF_j$. The stationary value that e_H assumes at points of $L_k \in PF_k$ equals $e_H(L_k) = -\lambda_k$. Calculation of the critical values of the energy is equivalent to finding the solutions λ of the eigenvalue problem $H(\psi) = \lambda\psi$. The minimum of e_H is necessarily a critical point, thus, the minimum is attained at points of PF_0 and the minimum value is $-\lambda_0$. If the multiplicity of $-\lambda_k$ is one then L_k is an isolated critical point. Proposition 2 then implies that L_k is non-degenerated, hence e_H is a Morse function if and only if all eigenvalues are simple. If $-\lambda_k$ has multiplicity greater than 1 the stationary set PF_k is a non-degenerated stationary manifold of positive dimension. Proposition 2 together with the Gradient Lemma allow the calculation of the index of PF_k. This index equals $\sum_{j<k}(\dim F_j - 1)$; this formula holds even if some F_j's are infinite dimensional.

21.-Radiation. The following terminology about smooth curves in PE will be adopted: Let $-\infty \leq t_0 < t_1 \leq +\infty$ and consider smooth curves $\Gamma : (t_0, t_1) \to PE$. The curve Γ *radiates* (resp. *absorbs*) if $e_H \circ \Gamma$ is strictly decreasing (resp. strictly increasing) on (t_0, t_1). Taking derivatives with respect to t this is seen to be equivalent to the conditions

$$\frac{d}{dt}(e_H \circ \Gamma) = \langle \nabla e_H(\Gamma(t)), \Gamma'(t)\rangle < 0 \tag{41}$$

$$\frac{d}{dt}(e_H \circ \Gamma) = \langle \nabla e_H(\Gamma(t)), \Gamma'(t)\rangle > 0 \tag{42}$$

for radiation and absorption respectively. Here the inner product is the one given by formula (40) if the scalars are real, or the real part of (40) if they are complex.

Consider now curves that are trajectories of the projective gradient flow $[\exp(-\kappa Ht)]$, that is, let $\Gamma(t) = [\exp(-\kappa Ht)]L$. By the very definition of generator of a flow and from the Gradient Lemma it follows that $\Gamma'(t) =$

$X_{-\kappa H}(\Gamma(t)) = -\frac{\kappa}{2}\nabla e_H(\Gamma(t))$. Therefore

$$\frac{d}{dt}\, e_H(\,[\exp(-\kappa H t)]\, L\,)\Big|_{t=t_0} = \frac{-\kappa}{2} \| \nabla e_H([\exp(\kappa H t_0)]\, L\,\|^2 < 0 \qquad (43)$$

and from condition (41), the energy function being the Rayleigh quotient, one obtains the

Lemma 2 (Radiation Lemma): *Non-constant trajectories of* $[\exp(-\kappa H t)]$ *radiate.*

If E is a complex Hilbert space E with associated complex projective space PE, from the way symplectic structures are defined on Kähler manifolds it follows that the Hamiltonian vector field of the energy function $\frac{1}{2}e_H : PE \to \mathbf{R}$ is $-\frac{i}{2}\nabla e_H$. The Gradient Lemma together with the linear dependency of X_H on H imply that $\frac{-i}{2}\nabla e_H = -iX_H = X_{-iH}$. This proves that X_{-iH} is a Hamiltonian field and is therefore energy conservative. The remarks made in section 3 apply then in the general infinite dimensional case. A detailed description of the trajectories of this Hamiltonian flow can be obtained from the PST. For a presentation of standard quantum mechanics using X_{-iH} see [Günther]. Compare also with [Gisin].

22.-Limit points and transition diagrams. The additional terminology to be introduced in this section is a standard or natural one, but will be explicitly stated for the sake of precision and clarity. It will be convenient to work initially with general dynamical systems rather than with the special case of projective systems. An introduction to dynamical systems in general, including an extensive discussion of limit points, can be found in [Bhatia-Szegö].

A *dynamical system with observable* is a system that consists of a manifold M, a flow $U_t : M \to M$ and a function $e : M \to \mathbf{R}$. These are the *space of states*, the *dynamics* and the *energy observable* respectively.

Let $U_t(x)$, $-\infty < t < \infty$, be a trajectory of the system. If the limits

$$\begin{array}{rcl} \alpha(x) & = & \lim_{t\to-\infty} U_t(x) \\ \omega(x) & = & \lim_{t\to\infty} U_t(x) \end{array} \qquad (44)$$

exist they are the α-limit and ω-limit, respectively, of the trajectory. These limit points are always stationary, that is, $U_t(\alpha(x)) = \alpha(x)$ and $U_t(\omega(x)) = \omega(x)$ for all t.

Let $\Gamma : (t_0, t_1) \to M$, $-\infty \leq t_0 < t_1 \leq \infty$, be a smooth curve in the space of states. The *energy change* along Γ is, by definition

$$\Delta\Gamma = \lim_{t \to t_1} e(\Gamma(t)) - \lim_{t \to t_0} e(\Gamma(t)) \tag{45}$$

provided the limits exist. If the curve Γ radiates then $\Delta\Gamma < 0$ and if Γ absorbs then $\Delta\Gamma > 0$.

The notion α and ω limits generalize as follows: The curve Γ *comes* from $W_0 \subseteq M$ if for any neighborhood N_0 of W_0 there exists τ_0, $\tau_0 > t_0$, such that $\Gamma(t) \in N_0$ for all $t_0 < t < \tau_0$. Similarly, Γ *goes* to W_1 if for any neighborhood N_1 of W_1 there exists τ_1, $\tau_1 < t_1$, such that $\Gamma(t) \in N_1$ for all $\tau_1 < t < t_1$. This applies in particular if W_0 and W_1 have unique elements, $W_0 = \{x_0\}$ and $W_1 = \{x_1\}$, in which case it will be said that Γ comes from x_0 and goes to x_1. If $t_0 = -\infty$ and $t_1 = \infty$ then for any $x = \Gamma(t)$, $\alpha(x) = x_0$ and $\omega(x) = x_1$.

For projective gradient systems the above limits can be calculated explicitly. For each eigenspace F_j let $Q_{F_j} : E \to E$ be the orthogonal projection over F_j. If $\psi_j = Q_{F_j}(\psi)$ then, because the eigenspaces span E, $\psi = \sum_{j=0}^{\infty} \psi_j$. For $\psi_j \neq 0$ define the *j-th projective component* of L as $L_j = [\psi_j]$; if $\psi_j = 0$ then then the j-th projective component is not defined. If k is the *smallest* of the indices j such that the j-th projective component of L is defined it will be proved below in the Transition Lemma that the ω-limit of L equals $\omega(L) = L_k$. Obviously, if the *energy ω-limit* is defined as $e_H^{\omega}(L) = \lim_{t \to \infty} e_H([\exp(-\kappa H t)]L)$ then $e_H^{\omega}(L) = e_H(L_k) = -\lambda_k$.

In case m is the *largest* index j such that the j-th projective component is defined then the *α-limit* of L exists and is $\alpha(L) = L_m$. Similarly, the *energy α-limit* is $e_H^{\alpha}(L) = \lim_{t \to \infty} e_H([\exp(-\kappa H t)]L) = e_H(L_m) = -\lambda_m$. In case $\psi_j \neq 0$ for infinitely many indices j the α-limit does not exist. But the energy α-limit still exists. This follows from the fact that, as a function of t, the energy $e_H[\exp(-\kappa H t)]L$ of the trajectory is bounded above by 0 but, for $t \to -\infty$, this function takes values larger than any of the $-\lambda_j$'s for which $\psi_j \neq 0$. Hence $e_H^{\alpha}(L) = 0$.

Lemma 3 (Transition Lemma): *Under the projective flow* $[\exp(\kappa Ht)]$ *for any state* $L \in PE$ *the* ω*-limit and energy* ω*-limit are* $\omega(L) = L_k$ *and* $e^{\omega}_H L = -\lambda_k$.

Proof: Define $F_L = \oplus F_j$, with the sum taken over the j's such that the projective component L_j is defined. This is the smallest H-invariant subspace of E that contains L. Then $L \in PF_L - P(\omega(L)^{\perp}) =$ total space of the vector bundle $\nu(\omega(L), F_L)$. The point L belongs to the fiber over $q(L) = Q_{\omega(L)}(L) = \omega(L)$. Apply now Theorem 1 to the restriction $A = -\kappa H|F_L$. Because all exponents $\kappa(\lambda_j - \lambda_k) = (\kappa\lambda_j + \kappa\lambda_k)$ in (23) are negative $\omega(L)$ equals the zero of the fiber over $\omega(L)$, namely, equals $\omega(L)$ itself. Then $e^{\omega}_H L = -\lambda_k$, QED.

The energy ω-limit is related to the quantum mechanical transition probabilities as follows. If the j-th projective component of L is defined, let

$$p_j = p_j(L) = \frac{|\langle \psi, \tilde{\psi}_j \rangle|^2}{\langle \psi, \psi \rangle \, \langle \tilde{\psi}_j, \tilde{\psi}_j \rangle} \tag{46}$$

and let $p_j = 0$ whenever $\psi_j = 0$. Thus, if $p_k \neq 0$ and $p_j = 0$ for all $j < k$ then $e^{\omega}_H L = -\lambda_k$. Therefore, the energy ω-limit is given by the smallest index k with non-zero transition probability $p_k \neq 0$. A more detailed analysis would show that if p_j is large then the trajectory of L is close to PF_j.

Corollary 1: *For* $-\lambda_k < -\lambda_m$ *there are states* $L \in PE$ *with* α*-limit in* PF_m *and* ω*-limit in* PF_k.

Proof: All states in $P(F_k \oplus F_{k+1} \oplus \cdots \oplus F_m)$ which are not in $PF_k \cup P(F_{k+1} \oplus \cdots \oplus F_m)$ have this property, QED.

Consider again a general dynamical system with observable. A state $x \in M$ is a *stationary state* and the energy value $-\lambda = e(x)$ is a *stationary energy level* if

$$\left. \frac{d\,(e(U_t(x)))}{dt} \right|_{t=0} = 0 \tag{47}$$

that is, if the derivative of e along the flow is zero at x. The *stationary set* $S \subseteq M$ is the collection of all stationary states and the *stationary value set* is the set $C = e(S) \subseteq \mathbf{R}$ of all the values taken by the energy function at

stationary states. For a stationary energy level $-\lambda$ let $S_{-\lambda} = \{x \in S | e(x) = -\lambda\} = e^{-1}(-\lambda) \cap S$ so that $S = \bigcup_{-\lambda \in C} S_{-\lambda}$, the union being disjoint. In case $S_{-\lambda}$ is a submanifold of M it will be called a *stationary manifold.* Obviously, for Hamiltonian dynamical systems the stationary value set is a continuum. See section 24 below.

Consider stationary states $x_k \in S_{-\lambda_k}$, $x_m \in S_{-\lambda_m}$. The curve Γ is a *transition* from x_m to x_k if it comes from x_m and goes to x_k. The notation $x_m \xrightarrow{U_t} x_k$ indicates that there exists at least one flow trajectory $\Gamma(t) = U_t(x)$ which is a transition from x_m to x_k, in other words, that there exists at least one $x \in M$ such that $U_t(x)$ has α-limit x_m and and ω-limit x_k. Similarly, since all states in the stationary set $S_{-\lambda_j}$ have energy $-\lambda_j$ the notation $-\lambda_m \xrightarrow{U_t} \lambda_k$ will be used to indicate that there exists at least one trajectory $\Gamma(t) = U_t(x)$ that comes from the stationary set $S_{-\lambda_m}$ and goes to the stationary set $S_{-\lambda_k}$.

The *transition diagram* of a dynamical system with observable consists of horizontal lines and vertical arrows and is constructed as follows: For each stationary level $-\lambda$ draw a horizontal line at height $-\lambda$, to be called *line* $-\lambda$. If at least one trajectory of U_t is a transition from stationary set $S_{-\lambda_m}$ to stationary set $S_{-\lambda_k}$, that is, if $-\lambda_m \xrightarrow{U_t} -\lambda_k$, then draw a vertical arrow from line $-\lambda_m$ to line $-\lambda_k$. This completes the construction. The transition diagram gives a conveniently simplified picture of the dynamical behavior of the observable e along the trajectories of the flow U_t.

23.-Transition diagrams for projective gradients. For a projective system with space of states PE, gradient flow $[\exp(-\kappa Ht)]$ and energy e_H the transition diagram is shown in Figure 1. For the specific case of the hydrogen atom the constant a appearing in the figure equals 13.6 electron-volts. The continuous energy observable e_H changes in time according how states $L \in PE$ evolve under the continuous and deterministic dynamics specified by the projective gradient evolution $[\exp(-\kappa Ht)]$. The diagram clearly shows that this evolution produces discretization of energy levels, in accordance with known physical properties of atoms. See the discussion of GE.4 in section 9 and see Corollary 1 in section 22 above for the existence of the transitions.

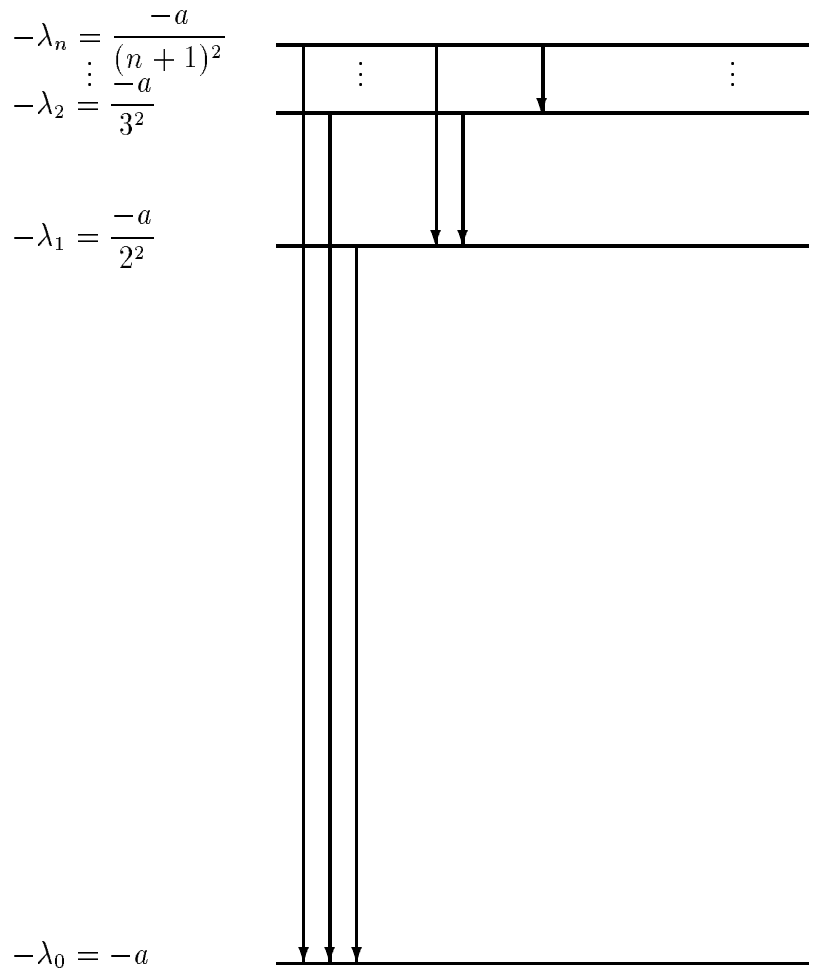

Figure 1. *Transition diagram for gradient system* $PE, X_{-\kappa H}, e_H$.

For a given operator H the transition diagram of the system with flow $[\exp(-\kappa Ht)]$ and energy observable e_H is independent of $\kappa > 0$. The role of κ is to determine the rate at which the system radiates energy. Only the speeds of trajectories are affected by κ. In other words, the curve $\Gamma(t)$ is a trajectory of $[\exp(Ht)]$ if and only if $\Gamma(\kappa t)$ is a trajectory of $[\exp(-\kappa Ht)]$.

24.-Transition diagrams for projective Hamiltonians. Consider now the projective system with Hamiltonian flow $[\exp(-iHt)]$ and energy e_H.

In this case the energy e_H is constant along trajectories and therefore all the values taken by the function e_H, that is, all elements of the interval $[-\lambda_0, 0)$, are stationary energy levels. These values fill a continuum and if stationary lines were drawn they would fill a solid rectangle. In contrast with gradients, Hamiltonians have an overabundance of stationary levels. Also, for Hamiltonians there are no transitions between different stationary levels because trajectories stay in constant energy hypersurfaces. Hamiltonian systems, again in contrast with gradients and in complete disagreement with the behaviour of atoms, have transition diagrams devoid of arrows.

The previous analysis and discussion confirm that the projective gradient systems with space of states PE, dynamics defined by the flow $[\exp(-\kappa Ht)]$ and energy observable e_H, is a good model for atomic systems. They constitute much more reasonable theoretical models than the Hamiltonian systems proposed by quantism. With the gradient evolution proposed by Realism the discretization of stationary energy levels and the existence of transitions are consequence of the dynamics. On the contrary, with the quantum Hamiltonian evolution the discretization of stationary energy levels and the transitions have to be postulated and contradict the dynamics.

25.-Energy packets. Energy emission by a projective gradient system will be analyzed now in more detail. The *energy radiated* up to time t by the trajectory $\Gamma(t) = [\exp(-\kappa Ht)]L$ with initial state $L = L^{(0)} \in PE$ is

$$G(t) = G_L(t) = -e_H([\exp(-\kappa Ht)]L) + e_H^\alpha(L) \tag{48}$$

Thus, $G(t) > 0$ and the conventional term $e_H^\alpha(L)$ in the formula ensures that $\lim_{t\to\infty} G(t) = 0$.

At each instant t the *energy output rate* of the trajectory of L is by definition the derivative $g(t) = g_L(t) = G'_L(t)$ of the radiated energy. This derivative will also be called the *energy packet* of the trajectory. The integral of the energy packet equals, up to sign, the energy change along the trajectory

$$\int_{-\infty}^{\infty} g(t)dt = \lim_{t\to\infty} G(t) - \lim_{t\to\infty} G(t) \tag{49}$$

A calculation gives

$$g(t) = 2\kappa(e_{H^2} - e_H^2)([\exp(\kappa Ht)]L) \tag{50}$$

and also

$$g(t) = \frac{\kappa}{2}\|\nabla e_H([\exp(-\kappa Ht)]L)\|^2 \tag{51}$$

Therefore it is natural to define the *radiance* $\rho(L)$ of a state L as

$$\begin{aligned} \rho(L) &= g_L(0) \\ &= 2\kappa(e_{H^2} - e_H^2)L \\ &= \tfrac{\kappa}{2}\|\nabla e_H(L)\|^2 \end{aligned} \tag{52}$$

the point being that if $L^{(1)} = [\exp(-\kappa Ht_1)]L$ then $\rho(L_1) = g_{L_1}(0) = g_L(t_1)$ meaning that in projective gradient systems the energy output rate is an explicit function of state.

The right hand term in equation (51) is just the middle term in formula (43) with reversed sign. But since $\omega(L) = L_k$=stationary state, equation (51) implies that

$$\lim_{t\to\infty} g(t) = 0 \tag{53}$$

Also, if $\alpha(L) = L_m$,

$$\lim_{t\to-\infty} g(t) = 0 \tag{54}$$

This last condition holds in all cases, even if the α-limit of L does not exist.

More generally, from continuity arguments and from formula (52) it follows that $g(t)$ becomes small for values of t such that $[\exp(\kappa Ht)]L$ is close to a stationary state. Therefore, if L has α-limit in PF_m and ω-limit in PF_k, the energy output rate $g(t)$ can be expected to have at most $m-k$ local maxima and $m-k-1$ local minima, depending on how near to $PF_j, k < j < m$, the trajectory of L passes. If L is not stationary and belongs to $P(F_k \oplus F_m)$ then $g(t)$ has a unique local maximum.

The energy output rate $g(t) = G'(t)$ has been defined as a function of time t. It can also be considered as a time dependent energy density, namely, as a differential form $g(t)\,dt = dG = G'(t)\,dt$. The physical assumption can be made now that the energy radiated propagates in space along an axis with constant direction and speed c. The differential form can be transformed to distance coordinate along an axis as follows. Let x be the distance along the

propagation axis so that $x = ct$ and $dx = c\,dt$. Defining the energy density in space variable x as

$$f(x) = \frac{1}{c} g(\frac{1}{c} x) \tag{55}$$

it follows that

$$f(x)\,dx = g(t)\,dt \tag{56}$$

If $x^{(0)} = ct^{(0)}$ then $f(x^{(0)})$ is the total energy contained in the (2-dimensional) plane perpendicular to the propagation axis at the point $x^{(0)}$ and time t_0. The function $f(x)$ is the *energy packet in axis variable* x.

26.-Energy absorption. Let $\Gamma : (t_0, t_1) \to PE$ be a smooth curve. Define the *driving speed* (of Γ, against $X_{-\kappa H}$) as

$$D(t) = \Gamma'(t) - X_{-\kappa H}(\Gamma(t)) \tag{57}$$

$D : (t_0, t_1) \to TPE$ is a vector field along Γ and $\Gamma' = X_{-\kappa H} + D$. Hence $D \equiv 0$ if and only if Γ is a trajectory of $X_{\kappa H}$. $D(t)$ is *zero at the end* if $\lim_{t \to t_1} D(t) = 0$. The driving speed provides a special argument to justify discretization of absorbed energy. It has more tautological than physical content.

Lemma 4 (Absorption Lemma): *Assume that* $\Gamma(t)$ *absorbs, goes to the set* $e_H^{-1}(-\lambda_k) = \{L | e_H(L) = -\lambda_k\}$ *and that the driving speed is zero at the end. Then* Γ *goes to* PF_k.

Proof: It suffices to prove that $\lim_{t \to t_1} X_{-\kappa H}(\Gamma(t)) = 0$, but this is a consequence of

$$\begin{array}{rcl} 0 & \leq & \lim_{t \to t_1} \langle X_{-\kappa H}(\Gamma(t)),\ X_{-\kappa H}(\Gamma(t)) \rangle \\ & = & \lim_{t \to t_1} \langle \Gamma'(t) - D(t),\ X_{-\kappa H}(\Gamma(t)) \rangle \\ & = & \lim_{t \to t_1} \langle \Gamma'(t), X_{-\kappa H}(\Gamma(t)) \rangle \\ & = & \lim_{t \to t_1} \langle \Gamma'(t), (-\frac{\kappa}{2}) \nabla e_H(\Gamma(t)) \rangle \\ & \leq & 0 \end{array} \tag{58}$$

The first inequality follows from the positivity of the inner product. Next, the first equality follows from the definition of $D(t)$, the second equality because $D(t)$ is zero at the end, the third equality from the Gradient Lemma and the last inequality is a consequence of equation (37). QED.

If interpreted physically this result tells that when the energy packet absorbed has the correct total energy and results in a trajectory with driving speed 0 at the end then a stationary state is reached. Up to a point, this argument justifies the existence of absorption spectra. The hypothesis that the driving speed is zero at the end depends on the interaction between the incoming energy packet and the system. A detailed physical description of the interaction requires, as said above, to set up and study a system in T^*PE. This will be done in a future paper.

27.-Stable manifolds and hyperbolicity. For a general discussion of stable and unstable manifolds and hyperbolicity of critical points and manifolds see [Abraham]. Also useful is [Bhatia-Szegö]. Consider again the projective flow $[\exp(-\kappa Ht)]$. If $C \subset PE$ is a set of critical points the *stable manifold* of C is

$$M^s(C) = \{L | \omega(L) \in C\} \tag{59}$$

and the *unstable manifold* is

$$M^u(C) = \{L | \alpha(L) \in C\} \tag{60}$$

Let

$$\begin{array}{rcl} F_k^- & = & \bigoplus_{j>k} F_j \\ F_k^+ & = & \bigoplus_{j<k} F_j \end{array} \tag{61}$$

so that $F_k^\perp = F_k^- \oplus F_k^+$. A point $L \in PE$ has $\omega(L) \in PF_k$ if and only if $L \in P(F_k \oplus F_k^-) - PF_k^-$ and from the Transition Lemma it follows that this is the stable manifold of PF_k under $[\exp(-\kappa Ht)]$. Reversing signs ($-\kappa H$ to κH) the same reasoning gives that the unstable manifold is $P(F_k \oplus F_k^+) - PF_k^+$. This proves the

Lemma 5 (Saddle Manifold Lemma): *The stable and unstable manifolds of* PF_k *under* $[\exp(-\kappa Ht)]$ *are*

$$M^s(PF_k) = P(F_k \oplus F_k^-) - PF_k^- \tag{62}$$

and

$$M^u(PF_k) = P(F_k \oplus F_k^+) - PF_k^+ \tag{63}$$

This Lemma is related to the hyperbolicity of the stationary manifold PF_k and tells exactly what the stable and unstable manifold of this stationary

manifold are. Note the relation with the classical 'saddle surface' that represents hyperbolic critical points of real valued functions of two variables. This can be further elaborated in terms of the second derivative of the energy e_H or, equivalently, in terms of the derivative of the projective vector field $X_{-\kappa H} = -\frac{\kappa}{2}\nabla e_H$.

Let F be a subspace of E with orthogonal complement in E equal to $F^\perp$, and consider a one-dimensional subspace L of F having orthogonal complement in F equal to $L^\perp \cap F$. Then the orthogonal complement of L in E splits as $L^\perp = (L^\perp \cap F) \oplus F^\perp$. If this is applied together with formula (31) then the tangent space to PE at L splits as $T_L PE = \mathrm{Hom}(L, L^\perp) = \mathrm{Hom}(L, L^\perp \cap F) \oplus \mathrm{Hom}(L, F^\perp) \simeq T_L PF \oplus T_L P(L \oplus F^\perp)$. This relation holds for all $L \in PF$. Substituting in equation (40) it can be proved that this direct sum of tangent spaces is actually an orthogonal sum. On the other hand, according to formula (34) $\bigcup_{L \in PF} \mathrm{Hom}(L, F^\perp)$ identifies canonically with $PE - PF^\perp =$ total space of $\nu(F, E)$. It follows that $\tau PE|PF = \tau PF \oplus \nu(F, E)$. Therefore $\nu(F, E)$ is the normal bundle of PF in PE.

Let now $F = F_k$. The sum $F_k^\perp = F_k^+ \oplus F_k^-$ gives

$$\tau PE|PF_k \cong \nu(F_k, F_k^-) \oplus \tau PF_k \oplus \nu(F_k, F_k^+) \tag{64}$$

Consider then the total space TPE of the tangent bundle τPE. The diagram

$$PE \stackrel{X_{-\kappa H}}{\longrightarrow} TPE \stackrel{q}{\longrightarrow} PE \tag{65}$$

can be differentiated and the result is a new diagram with objects the total spaces of the tangent bundles of the corresponding manifolds and with arrows the bundle derivatives of the respective maps, as shown immediately below

$$TPE \stackrel{TX_{-\kappa H}}{\longrightarrow} TTPE \stackrel{Tq}{\longrightarrow} TPE \tag{66}$$

The states belonging to the stationary manifold PF_k are zeros of the projective vector field $X_{-\kappa H}$, therefore the composition $Tq \circ TX_{-\kappa H}$ of derivatives is a well-defined bundle endomorphism B on the restriction $\tau PE|PF_k$ of the tangent bundle τPE to the stationary manifold PF_k. According to the proof of the Projective Spectral Theorem for $L_k \in F_k$ the vector field $X_{-\kappa H}$ is linear on the fibers of $\tau PE|PF_k$, hence, $X_{-\kappa H}$ has derivative equal to that linear

map. Hence the restriction B_{L_k} of B to the fibre $T_{L_k}PE = \mathrm{Hom}(L_k, L_k^\perp) \cong L_k^\perp$ equals the linear endomorphism $-\kappa(H - \lambda_k I) : L_k^\perp \to L_k^\perp$. In particular the eigenvalues of B_{L_k} are independent of $L_k \in PF_k$, and formula (64) is a splitting by subbundles that are invariant under the projective flow. The eigenvalues of B are negative, zero and positive in $\nu(F_k, F_k^-)$, τPF, and $\nu(F_k, F_k^+)$ respectively, being of the form $\kappa\lambda_j - \kappa\lambda_k$. Therefore formula (64) is actually a hyperbolic splitting of τPE along PF_k. This proves that the critical manifolds are hyperbolic.

This hyperbolicity of $X_{-\kappa H}$ at PF_K plays an key role in Realism. It is implicit in the Perturbation Lemma and Saddle Manifold Lemma of the next section. Hyperbolicity explains the spontaneous transition phenomena of microscopic physical systems. The apparent randomness usually associated to these transitions is explained by Realism as the deterministic evolution a of perturbed stationary state. The almost stationary states evolve with sensitive dependency on initial conditions. The sensitivity is due to the hyperbolic nature of the projective flow near the stationary manifolds.

28.-Perturbations. Perturbation theory is a classical topic with numerous ramifications. Again, [Abraham] is a good reference. The Perturbation Lemma below tells how gradient projective systems behave when a stationary state is perturbed. The process under discussion consists of an initial stationary state L_m which is perturbed to a nearby state $\tilde{L}_m$ and then evolves as a trajectory $[\exp(-\kappa Ht)]\tilde{L}_m$ of the projective gradient system. For example, it can be concluded from the Perturbation Lemma that the total energy change is always an eigenvalue difference $\lambda_m - \lambda_k$. The techniques to be used are standard ones in Morse Theory.

Let

$$W_m = e_H^{-1}[-\lambda_0, -\lambda_{m+1}) = \{L \in PE | e_H(L) < \lambda_{m+1}\} \tag{67}$$

This set is open in PE, contains PF_j for all $j < m + 1$, and because of the Radiation Lemma is a set invariant under the projective flow $[\exp(-\kappa Ht)]$. Consider a neighborhood N_m of $L_m \in PF_m$ such that $N_m \subset W_m$. Define

$$N_m^- = \{L \in N_m - PF_m | \omega(L) \in PF_m\} \tag{68}$$

$$N_m^0 = N_m \cap PF_m \tag{69}$$

$$N_m^+ = \{L \in N_m | \omega(L) \notin PF_m\} \tag{70}$$

Then $N_m = N_m^- \cup N_m^0 \cup N_m^+$ is a disjoint union. Note that $N_m^+ = \emptyset$ if and only if $m = 0$.

Lemma 6 (Perturbation Lemma): *States $\tilde{L}_m$ close enough to PF_m satisfy one and only one of the following conditions:*

a) $\tilde{L}_m$ is in PF_m.
b) $\tilde{L}_m$ is not in PF_m and has energy ω-limit $e_H^\omega \tilde{L}_m = -\lambda_m$.
c) $\tilde{L}_m$ is not in PF_m and has energy ω-limit $e_H^\omega \tilde{L}_m = -\lambda_k$ for some $k < m$.

Proof: This follows from the definitions of W_k, of energy ω-limit and from the Transition Lemma, QED.

29.-Canonical bundles and tangent bundle to PE. In [Milnor-Stasheff] the equivalence of the tangent bundle τPE with the bundle $\mathrm{Hom}(\gamma, \gamma^\perp)$ is proved for finite dimensions. The proof is presented here applies in the infinite dimensional case. The result is a preliminary for the Gradient Lemma.

The *canonical bundle* $\gamma = \gamma_E$ over PE is the line bundle with total space

$$T\gamma = \{(L, \psi) \in PE \times E \,|\, \psi \in L\} \tag{71}$$

and projection $q_\gamma(L, \psi) = L$. Each $L_0 = [\psi_0] \in PE - \{0\}$ defines a local trivialization of γ over $V_0 = PE - PL_0$ given by the diagram where $\hat{k}(L, \psi) = (L, Q_{L_0}(\psi)$. The maps pr_j, $j = 1, 2$, with domain the product $V_0 \times L_0$ are natural projections onto the first and second factor respectively.

The *orthogonal canonical bundle* $\gamma^\perp = \gamma_E^\perp$ over PE has fibers which are hyperplanes of E and has total space

$$T\gamma^\perp = \{(L, \eta) \in PE \times E | \eta \perp L\} \,. \tag{72}$$

The projection is $q_{\gamma^\perp}(L,\eta) = L$. Local trivializations over V_0 are defined by

$$\begin{array}{ccc} q_{\gamma^\perp}^{-1}(V_0) & \xrightarrow{\hat{k}_0^\perp} & V_0 \times L_0^\perp \\ & q_{\gamma^\perp} \searrow \quad \swarrow \; pr_1 & \\ & V_0 & \end{array} \tag{73}$$

with $\hat{k}_0^\perp(L,\eta) = (L, Q_{L_0}(\eta))$.

The vector bundle $\tau = \mathrm{Hom}(\gamma, \gamma^\perp)$ over PE has fiber $\mathrm{Hom}(L, L^\perp)$ over $L \in PE$, hence total space

$$T_\tau = \bigcup_{L \in PE} \mathrm{Hom}(L, L^\perp) \tag{74}$$

and projection $q_\tau : T_\tau \to PE$ that sends points in $\mathrm{Hom}(L, L^\perp)$ to L. Local trivializations over open affine sets V_0 are provided by

$$\begin{array}{ccc} q_\tau^{-1}(V_0) & \xrightarrow{\hat{h}_0} & V_0 \times L_0^\perp \\ & q_\tau \searrow \quad \swarrow \; pr_1 & \\ & V_0 & \end{array} \tag{75}$$

with $\hat{k}_0^\perp(L,\eta) = (L, Q_{L_0}(\eta))$. The top arrow is defined as follows: for $\theta \in \mathrm{Hom}(L, L^\perp) \subset q_\tau^{-1}(V_0)$

$$\hat{h}_0(\theta) = (L, \hat{k}_0^\perp \circ \theta \circ pr_2 \circ \hat{k}_0^{-1}(L, \psi_0)) \tag{76}$$

Here $\hat{k}_0$ and $\hat{k}_0^\perp$ are the corresponding local trivializations of the canonical and orthogonal canonical bundles.

It will now be verified that the tangent bundle to PE is canonically isomorphic to the bundle $\tau = \mathrm{Hom}(\gamma, \gamma^\perp)$ above. To verify this, let $\psi_0, \psi_1 \in E - \{0\}$ and consider the commutative diagram

$$V_1 \times L_1^\perp \qquad\qquad V_1 \times L_1^\perp$$

$$
\begin{array}{ccccccccc}
 & h_1^{-1}\times \mathrm{id}\swarrow & \searrow (\hat{k_1}^{\perp})^{-1} & & & & \hat{k_0}^{\perp}\swarrow & \searrow h_0\times \mathrm{id} & \\
L_1^{\perp}\times L_1^{\perp} & \longleftarrow & q_\tau^{-1}(V_1) & \longleftarrow & q_\tau^{-1}(V_0\cap V_1) & \longrightarrow & q_\tau^{-1}(V_0) & \longrightarrow & L_0^{\perp}\times L_0^{\perp} \\
pr_1\downarrow & & q\downarrow & & q\downarrow & & q\downarrow & & pr_1\downarrow \\
L_1^{\perp} & \overset{h_1^{-1}}{\longleftarrow} & V_1 & \longleftarrow & V_0\cap V_1 & \longrightarrow & V_0 & \overset{h_0}{\longrightarrow} & L_0^{\perp}
\end{array}
\tag{77}
$$

The lower edges of the diagram triangles are defined by the condition of that the triangles should be commutative, and the remaining unnamed arrows are inclusions. The first two arrows in the top horizontal line of the diagram, suitably restricted, have inverses which composed with the two last arrows in the same line define a map $\Phi : h_1(V_0 \cap V_1) \times L_1^{\perp} \to h_0(V_0 \cap V_1) \times L_0^{\perp}$ and define also a commutative diagram

$$
\begin{array}{ccc}
h_1(V_0\cap V_1)\times L_1^{\perp} & \overset{\Phi}{\longrightarrow} & h_0(V_0\cap V_1)\times L_0^{\perp} \\
pr_1\downarrow & & pr_1\downarrow \\
h_1(V_0\cap V_1) & \overset{h_0\circ h_1^{-1}}{\longrightarrow} & h_0(V_0\cap V_1)
\end{array}
\tag{78}
$$

A routine calculation can now be performed to show that

$$
\Phi(\eta,\eta') = (h_0\circ h_1^{-1}(\eta), D(h_0\circ h_1^{-1})(\eta)\cdot\eta') \tag{79}
$$

It has therefore been proved that when expressed in the projective coordinate system the local trivializations of $\mathrm{Hom}(\gamma,\gamma^{\perp})$ have transition functions given by the derivative of $h_0\circ h_1^{-1}$. But these are the transition functions of the tangent bundle and this implies that $\tau PE \cong \tau = \mathrm{Hom}(\gamma,\gamma^{\perp})$.

30.-Proof of the Gradient Lemma. Smooth maps $f : PE \to \mathbf{R}$ have bundle derivatives that are bundle maps

$$
\begin{array}{ccc}
TPE & \overset{Tf}{\longrightarrow} & \mathbf{R}\times\mathbf{R} \\
q_\tau\downarrow & & \downarrow pr_1 \\
PE & \overset{f}{\longrightarrow} & \mathbf{R}
\end{array}
\tag{80}
$$

The tangent bundle τPE has just been identified with $\mathrm{Hom}(\gamma, \gamma^\perp)$. Under this identification the map Tf is given locally as follows: For $\theta \in \mathrm{Hom}(L, L^\perp) \subseteq q_\tau^{-1}(V_0)$

$$Tf(\theta) = (f(L), D(f \circ h_0^{-1})(h_0(L)) \cdot pr_2 \circ \hat{h}_0(\theta)) \ . \tag{81}$$

Let $\psi \in L \in PE$ and consider the vector field defined on PE by

$$X_A(L)\psi = Q_L(A(\psi)) \tag{82}$$

Here X_A is considered as a section of the bundle $\mathrm{Hom}(\gamma, \gamma^\perp)$. It is immediate that the zeros of X_A are the one-dimensional eigenspaces of A. Over a coordinate system centered at $L_k \in PF_k$ the local trivialization of the tangent bundle defined by formula (55) (with $\psi_0 = \psi_k \in L_k \in PF_k$) transforms X_A into the linear vector field $(A - \lambda_k I) : L_k^\perp \to L_k^\perp$. This is an easy calculation. Because the eigenvectors of A span E, projective coordinate systems centered at critical points cover PE. From Propositions 2 and 3 it follows then that X_A equals over all of PE to the infinitesimal generator of the flow $[\exp(At)]$ defined in (13). This completes the proof of (19).

Because X_A depends linearly on A and e_H is linear in H, it suffices to prove (26) in the case $\kappa = 1$. Consider first real scalars, and let $L_0 \in PE$, $\psi_0 \in L_0$ a unit vector. If $Te_H : \mathrm{Hom}(L, L^\perp) \to \mathbf{R}$ is the bundle derivative of e_H at L_0, the Gradient Lemma is equivalent to the following relations:

$$\begin{array}{rcll} \langle X_{-H}(L_0), \theta \rangle_{\mathrm{Hom}(L_0, L_0^\perp)} & = & \frac{1}{2} Te_H(L_0) \cdot \theta & \\ 2\langle Q_{L_0^\perp} H(\psi_0), \theta(\psi_0) \rangle_E & = & Te_H(L_0) \cdot \theta \ , & \theta \in \mathrm{Hom}(L_0, L_0^\perp) \ . \end{array} \tag{83}$$

The last term can be calculated using the projective coordinate system $h_0 : V_0 \to L_0^\perp$ defined by ψ_0 and centered at L_0. Note that L_0 is not assumed to be an eigenvector of H. Applying now (39)

$$Te_H(L_0)(\theta) = (e_H(L_0), D(e_H \circ h_0^{-1})(h_0(L_0)) \cdot pr_2 \hat{h}_0(\theta)) \ . \tag{84}$$

On the other hand,

$$\begin{array}{rcl} h_0(L_0) & = & 0 \\ pr_2 \hat{h}_0(\theta) & = & Q_{L_0^\perp}(\theta(\psi_0)) \end{array} \tag{85}$$

and

$$e_H \circ h_0^{-1}(\eta) = \langle H(\psi_0+\eta), \psi_0+\eta\rangle / \langle \psi_0+\eta, \psi_0+\eta\rangle \ . \tag{86}$$

Elementary computations can then be performed to conclude that

$$\begin{aligned} D(e_H \circ h_0^{-1})(0)\cdot(Q_{L_0^\perp}(\theta(\psi_0)) &= 2\langle \psi_0, Q_{L_0^\perp}(\theta(\psi_0))\rangle \\ &= 2\langle Q_{L_0^\perp}(\psi_0), \theta(\psi_0)\rangle \ . \end{aligned} \tag{87}$$

This completes the proof in the real case. For complex Hilbert spaces the real part of equation (35) is used and the proof reduces to the real case.

31.-Real scalars vs. complex scalars. In Realism wave functions ψ should be real valued functions. This makes E a real vector space and PE a real projective space. This choice of scalars is necesary to make angular momentum correspond to an actual rotation of the wave function. The dynamics of rotations is defined by means of a skew-adjoint operator K that commutes with the self adjoint Hamiltonian H and is added to H to define rotational dynamics. A real normal operator $H+K$ results. If complex functions are used then stationary wave functions in the eigenspaces F_k differing in a space rotation are equal except for a complex phase of the form $e^{i\theta}$. Therefore at the level of the complex projective space PE they become identical states. The degree of freedom necessary to describe wave rotation is lost. In consequence, quantism requires Spin matrices to refer in an extremely akward way to rotational phenomena. The quantum Spin theory is a complicated and unclear description of a basic microscopic physical phenomenon: The physical rotation of waves.

For example, according to Realism, Zeeman effect is physically a phenomenon of wave rotation described by a flow $[\exp(-(H+K)t)]$ on real projective space, with H=self-adjoint Hamiltonian and K=rotation operator, both defined on the real Hilbert space $L^2_{\mathbf{R}}(\mathbf{R}^3)$ of real valued, square integrable functions defined on $\mathbf{R}^3$. The dynamics of this flow is more involved than the one given in Theorem 1 since it deals simultaneously with gradient evolution *and* rotations of wave functions. A projective spectral theorem for this flow can be found in [3] or [4].

REFERENCES

[1] D. Crespin, *Realism* Preprint, Oct. 1992. Privately circulated. An electronic version is available at URL below.
[2] D. Crespin, *Stability of dynamical systems and Quantum Mechanics.* Notas de Matemáticas (83-TP-01), Facultad de Ciencias UCV, 1983.
[3] D. Crespin, *Projective spectral theorems and deterministic atoms.* Acta Científica Venezolana 38:570-574, 1987.
[4] D. Crespin, *Projective spectral theorem for real normal operators and electron rotation.* To appear at URL below.
[5] D. Crespin, *Projective Dynamical Systems and Realism.* The present paper. This reworked version of [2] is available at URL below.
[Abraham, R. and Marsden, J.]: *Foundations of Mechanics*; 2nd ed.; Benjamin/Cummings, 1978.
[Bhatia-Szegö]: Stability Theory of Dynamical Systems; Springer-Verlag, 1970.dc
[Gisin, N.]: A Model of Dissipative Quantum Dynamics. Preprint, 1982.
[Günther, C.]: Prequantum bundles and projective Hilbert geometries, *Int. J. of Theor. Phys.* **16, No.6**, 1977, 447–467.
[Milnor, J.W.]: *Morse Theory* Ann. of Math. Studies, **51**, Princeton University Press, 1968.
[Milnor, J.W. and Stasheff, J.D.]: *Characteristic Classes; Ann. of Math. Studies* **76**, Princeton University Press, 1974.
[Morse, M. and Cairns, S.S.]: *Critical Point Theory in Global Analysis and Topology*; Academic Press, 1969.

Daniel Crespin
Escuela de Física y Matemáticas
Facultad de Ciencias
Universidad Central de Venezuela.

dcrespin@euler.ciens.ucv.ve
http://euler.ciens.ucv.ve/Professors/dcrespin/Pub/

Galipán, 31 de Marzo de 1996.