INERTIA GROUPS OF MANIFOLDS

A Dissertation

Presented to

The Faculty of The Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

In Partial Fulfillment

of the Requirements of the Degree

Doctor of Philosophy

By

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December 1971

NOTE: This PhD thesis was typewritten in 1971. Brandeis University Library filed it under 'BRYDEN, Daniel Crespin' instead of 'CRESPIN-BRYDEN, Daniel' or 'CRESPIN, Daniel'. Why?: In most Spanish speaking countries legal names always include both father's family name and mother's family maiden name, a fact I failed to warn librarians about; they assumed Crespin was a given name and Bryden the family name.

Besides being shelved at a Brandeis library, the thesis was microfilmed but remained otherwise unpublished. The present LATEX version was transcribed in December 2007, thirty six years afterward. Some gaps in the original typing have been filled, and some minor corrections —including many misprints— were done. The numbering of sections, Theorems, Propositions, etc. has been preserved. Page numbering, however, has changed.

At the time the thesis was written Differential Topology was most fashionable, but nowadays many other subjects dispute the mathematical heavyweight championship.

ABSTRACT

INERTIA GROUPS OF MANIFOLDS

(A Dissertation presented to the Faculty
of the Graduate School of Arts and Sciences
of Brandeis University, Waltham, Massachusetts.)

by

Daniel Crespin Bryden

Inertia and concordance inertia groups of smooth manifolds are studied.

Geometric constructions, an invariant f_R of Brumfiel and certain invariant g_R deduced from it are used.

It is proved that for any oriented smooth manifold M^{4n-1}, the concordance inertia group $I_c(M^{4n-1})$ is a proper subgroup of Θ_{4n-1}, in fact, it does not contain the Milnor sphere.

Then a certain pairing $\bar{\rho}_{n,k-1}$ with domain $\Theta_n \times \pi_{n+k-1}(S^{n-1})$ and values in Θ_{n+k} is constructed and related to concordance inertia groups.

There is always a manifold W^{n+k} having concordance inertia group containing the image of $\bar{\rho}_{n,k-1}$. We prove also in a number of cases that the image of $\bar{\rho}_{n,k-1}$ is away from bP_{n+k+1}.

The author acknowledges the guidance and encouragement of his thesis adviser Dr. Jerome Levine.

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Introduction.

In this work the inertia groups and the concordance inertia groups of manifolds are studied. This has been done previously in [Browder], [Brumfiel 3], [Kosinski], and [Munkres 2]. Most results obtained here are about concordance inertia groups. The techniques used are based on geometric arguments and on the results of [Brumfiel 1, 2, 3]. The work is organized as follows.

In Chapter 1 preliminary material is presented. This consists of the following: First, some well known properties of homotopy spheres. Then a brief review of characteristic classes and numbers. Next a description of Brumfiel's theorems and his invariants, which are basic in this work. Finally, a theorem of Anderson-Brown-Peterson is stated and Bredon's pairing is described.

In Chapter 2 an integrality theorem —Theorem 6— for the index of certain manifolds is proved using Brumfiel's and Anderson-Brown-Peterson's results. By means of Theorem 6 a certain invariant g_R becomes computable in many cases. Related results are proved.

In Chapter 3 piecewise differentiable (abbreviated PD) maps are discussed. Next, inertia groups are defined and i-diffeomorphisms (see page 21) are proved to exist. We give then upper bounds for the inertia groups of certain manifolds (Proposition 14). Then concordance inertia groups are defined and upper bounds are given for them. In particular it is proved that for any closed orientable manifold M^{4n-1} the Milnor sphere, Σ_0^{4n-1}, is not in the concordance inertia group $I_c(M^{4n-1})$.

In Chapter 4 a pairing $\bar{\rho}_{n,k}$ is constructed. This is closely related to Bredon's pairing $\rho_{n,k}$, and to Milnor-Munkres-Novikov pairing $\tau_{n,k}$. In Theorem 25 a relationship is obtained between the image of $\bar{\rho}_{n,k}$ and the concordance inertia group of a manifold W^{n+k}, the relationship being homotopy theoretical. From this it follows (Theorem 26) that the image of $\bar{\rho}_{n,k}$ is always contained in the concordance inertia group of some manifold. Finally, applying results of Chapter 2, conditions on the image of $\bar{\rho}_{n,k}$ are obtained which imply that for certain values of n and k the images of $\bar{\rho}_{n,k}$ and of $\rho_{n,k}$ can be described completely in homotopy theoretical terms.

Chapter 5, where proposition 18 is proved, is mainly technical.

Chapter 6 presents some low dimensional calculations.

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0.- NOTATION.

The notation used is standard:

$\mathbb{R}^n$ denotes n-dimensional Euclidean space.

$D^n \subseteq \mathbb{R}^n$ is the n-dimensional unit ball.

rD^n is the n-dimensional ball of radius r.

$S^{n-1} \subseteq \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere.

$rS^{n-1} \subseteq \mathbb{R}^n$ is the $(n-1)$-sphere of radius r.

$D^{n-1}_+ = \{(x_1, \ldots, x_n) \in S^{n-1} | x_n \geq 0\}$

$D^{n-1}_- = \{(x_1, \ldots, x_n) \in S^{n-1} | x_n \leq 0\}$

If X and Y are topological spaces, denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

If X is a path connected space with base point $x_0 \in X$, then $\pi_n(X)$ is the n-th homotopy group of X.

$SO(n)$ is the rotation group of $\mathbb{R}^n$, $SO(n) \subseteq SO(n+1)$ and

$$SO = \bigcup_{n=1}^{\infty} SO(n)$$

$J = J_{k,n} : \pi_k(SO(n)) \to \pi_{n+k}(S^n)$ is the J-homomorphism (see [Kervaire]). If $\pi_k^S = \text{lím}_n\, \pi_{n+k}(S^n)$ there is defined a J-homomorphism

$$\pi_k(SO) \to \pi_k^S$$

Let $\text{Coker}(J_k) = \pi_k^S / \text{Image}(J_k)$.

t_n is the integer

$$t_n = 2^{2n-2}(2^{2n-1} - 1) \cdot a_n \cdot \text{numerator} \left(\frac{B_n}{4n} \right)$$

where B_n is the n-th Bernoulli number, $a_n = 1$ for even n and $a_n = 2$ for odd n; see [Levine 2], page 22.

Let t'_n be the largest odd number that divides t_n. It follows from a theorem of Von-Staudt (cf. [Levine 1]) that

$$t'_n = (2^{2n-1} - 1) \cdot \text{numerator} \left(\frac{B_n}{4n} \right)$$

The term 'piecewise linear' will be abbreviated 'PL'. For the definition of PL isomorphism, PL ball, and other piecewise linear concepts see [Hudson].

All manifolds considered here are orientable, however, orientability will often be stated explicitly.

References are made to author, with a number when necessary. For example [Levine 2] refers to the second paper by Levine listed in the bibliography.

1. Preliminary material

1.1.- Homotopy spheres.

Denote by Θ_n the group of h-cobordism classes of oriented smooth manifolds which are homotopy equivalent to the sphere S^n, with the group operation induced by connected sum.

Denote by $\Theta_n(1)$ the group of oriented diffeomorphism classes of smoothings of S^n, the group operation is induced by connected sum.

Denote by $\Theta_n(2)$ the group of concordance classes of smoothings of S^n with the group operation induced by connected sum.

Denote by $\Theta_n(3)$ the group of concordance classes of orientation preserving diffeomorphisms from the sphere S^{n-1} to itself under the operation induced by composition of diffeomorphisms.

Denote by $\Theta_n(4)$ the group of concordance classes relative to the boundary of orientation preserving diffeomorphisms $f : D^{n-1} \to D^{n-1}$ which are the identity on a neighborhood of the boundary $\partial D^{n-1} = S^{n-2}$. The group operation is induced by composition.

Homomorphisms that relate these various groups will be now described.

The obvious map $\Theta_n(1) \to \Theta_n$ is a group homomorphism.

The map that assigns to a diffeomorphism class of smoothings the concordance class of any of its representatives induces a well defined group homomorphism $\Theta_n(1) \to \Theta_n(2)$.

For a homomorphism $\Theta_n(3) \to \Theta_n$ proceed as follows. If $f : S^{n-1} \to S^{n-1}$ is a diffeomorphism let Σ_f be the homotopy sphere obtained by gluing two copies of the n disk D^n along their boundaries via the diffeomorphism f. Then $f \to \Sigma_f$ induces a well defined group homomorphism.

Finally, given a diffeomorphism $g : D^{n-1} \to D^{n-1}$ which is the identity on some neighborhood of the boundary, let $\lambda : D^{n-1} \to S^{n-1}$ be the embedding given by

$$\lambda(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, \sqrt{1 - (x_1^2 + \cdots + x_{n-1}^2})$$

and define

$$\bar{g}(y) = \begin{cases} \lambda \circ g \circ \lambda^{-1}(y) & \text{if } y \in \lambda(D^{n-1}) \\ y & \text{if } y \notin \lambda(D^{n-1}) \end{cases}$$

The correspondence $g \to \bar{g}$ induces a well defined group homomorphism $\Theta_n(4) \to \Theta_n(3)$. All these group homomorphisms are isomorphisms in case

$n \geq 5$. For more details see [Smale].

Recall now some well known facts about the structure of the group Θ_n.

Θ_n is an abelian group, finite if $n \neq 3$. Θ_3 is unknown.

Let bP_{n+1} consist of those homotopy n-spheres which bound some parallelizable $(n+1)$-manifold, then:

For $n \neq 1$, bP_{4n} is cyclic of order t_n (see page 5) generated by an element Σ_0^{4n-1} called the Milnor sphere, which bounds a parallelizable manifold of index 8.

bP_{2n+2} is cyclic of order at most 2. The order is 2 if $n \neq 2^i - 2$. The order is 1 if $n = 0,\ 2,\ 6,\ 14$ or 30. The remaining cases are unsettled. In particular bP_{8n+2} is cyclic of order 2; it has a generator Σ_0^{8n+1} called the Kervaire sphere, which bounds a parallelizable manifold of Arf invariant 1.

bP_{2n+1} is trivial for all n.

For $n \neq 2^i - 2$ there is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \xrightarrow{p'} \mathrm{Coker} J_n \to 0$$

Here p' is induced by the Thom-Pontriagin construction. For more details see [Kervaire-Milnor] and [Levine2].

1.2.- The signature, characteristic classes and characteristic numbers.

Let M^{4n} be an oriented closed manifold with orientation class $[M^{4n}] \in H_{4n}(M^{4n};\mathbb{Q})$. On $H^{2n}(M^{4n};\mathbb{Q})$ consider the quadratic form defined by cup product

$$(x, y) \to \langle x \smile y, [M^{4n}] \rangle$$

The signature, or index, of M^{4n}, $s(M^{4n})$, is the index of this bilinear form.

This is an oriented cobordism invariant and

$$s(M_1^{4n} \# M_0^{4n}) = s(M_1^{4n}) - s(M_0^{4n})$$

Let M^{4n} be oriented with possibly non-empty boundary, ∂M^{4n}, and orientation class $[M^{4n}] \in H_{4n}(M^{4n}, \partial M^{4n};\mathbb{Q})$. On $H^{2n}(M^{4n}, \partial M^{4n};\mathbb{Q})$ consider the quadratic form $q : (x, y) \to \langle x \smile y, [M^{4n}] \rangle$. The one can extend the previous definition to the relative case by setting $s(M^{4n})$ =signature of q.

Let M_1^{4n}, M_0^{4n} be compact smooth manifolds and $f : \partial M_1^{4n} \to \partial M_0^{4n}$ be an orientation preserving diffeomorphism. Denote by $M_1^{4n} \bigcup_f (-M_0^{4n})$ the manifold obtained from the disjoint union $M_1^{4n}+M_0^{4n}$ by identifying $x \in M_1^{4n}$ and $f(x) \in M_0^{4n}$. Give to $M_1^{4n} \bigcup_f (-M_0^{4n})$ the orientation induced from M_1^{4n}. One then has ([Sullivan], p. 6.11)

$$s(M_1^{4n} \bigcup_f M_0^{4n}) = s(M_1^{4n}) - s(M_0^{4n})$$

Also, if $M_1^{4n} \coprod M_0^{4n}$ denotes the connected sum along the boundary (see [Kervaire-Milnor]) then $s(M_1^{4n} \coprod M_0^{4n}) = s(M_1^{4n}) - s(M_0^{4n})$.

Let ξ be a vector bundle over a CW complex X. Then the i-th Stiefel-Whitney class of ξ, $w_i(\xi) \in H^i(X;\mathbb{Z}_2)$, is defined.

If M^n is a compact smooth manifold with tangent vector bundle τ_{M^n}, the i-th Stiefel-Whitney class of M^n is $w_i(M^n) = w_i(\tau_{M^n}) \in H^i(M^n;\mathbb{Z}_2)$. Note that w_i is defined for manifolds with possibly non-empty boundary.

It is possible to extend the definition of w_i to include all closed topological manifolds, in particular to include closed PL manifolds. This is done using the Wu classes and Steenrod squares. Recall that a smooth or PL manifold M^n is spin if it is orientable and if $w_2(M^n) = 0$.

Let η be an oriented vector bundle over a CW complex X. Then the i-th rational Pontriagin class of η, $p_i(\eta) \in H^{4i}(X;\mathbb{Q})$, is defined.

If M^n is a compact oriented smooth manifold with oriented tangent vector bundle τ_{M^n} then the i-rational Pontriagin class of M^n is $p_i(\tau_{M^n}) \in H^{4i}(M^n;\mathbb{Q})$. The rational Pontriagin classes are defined for smooth manifolds with possibly non-empty boundary, and they can be defined on all closed PL manifolds using Hirzebruch signature theorem (see [Milnor]).

Let M^{4n} be a closed oriented (PL or smooth) manifold with orientation class $[M^{4n}] \in H^{4n}(M^{4n};\mathbb{Q})$. Consider partitions of n, i.e. finite sequences $\omega = (i_1,\ldots,i_r)$ of positive integers such that $i_1+\cdots+i_r = n$. Then there is defined a corresponding Pontriagin number of M^{4n}

$$p_\omega(M^{4n}) = p_{i_1\cdots i_r}(M^{4n}) = \langle p_{i_1}(M^{4n}) \smile \cdots \smile p_{i_r}(M^{4n}), [M^{4n}]\rangle \in \mathbb{Q}$$

This rational number is an integer if the manifold M^{4n} is smooth. These numbers are invariant under smooth oriented cobordism. Also, if two closed PL manifolds are PL cobordant (in particular if they are isomorphic) then they have the same rational Pontriagin numbers. Recall that a Pontriagin number $p_{i_1\cdots i_r}(M^{4n})$ is decomposable if $r \geq 2$. Note that if M^{4n} is a smooth compact manifold with non-empty boundary, the above definition is vacuous since $H_{4n}(M^{4n};\mathbb{Q})$ is trivial. However, following [Ells and Kuiper] it is possible to define decomposable Pontriagin numbers for certain manifolds with non-empty boundary. This is done as follows: Suppose M^{4n} is a smooth oriented manifold with boundary ∂M^{4n}. If for $0 \leq i \leq n-1$ the homomorphisms

$$j^* : H^{4i}(M^{4n}, \partial M^{4n};\mathbb{Q}) \to H^{4i}(M^{4n};\mathbb{Q})$$

induced by the inclusion of pairs $j : (M^{4n}, \emptyset) \to (M^{4n}, \partial M^{4n})$ are isomorphisms, define the relative Pontriagin numbers of M^{4n} as

$$p_{i_1\cdots i_r}(M^{4n}) = \langle j^{*\,-1}(p_{i_1}(M^{4n})) \smile \cdots \smile j^{*\,-1}(p_{i_r}(M^{4n})), [M^{4n}]\rangle \in \mathbb{Q}$$

where $[M^{4n}]$ is the orientation class of M^{4n} in $H^{4n}(M^{4n}, \partial M^{4n};\mathbb{Q})$. This definition of Pontriagin numbers extends the previous one and applies when ∂M^{4n} is a homotopy sphere.

These relative Pontriagin numbers have the following additive property: Let M_1^{4n}, M_2^{4n} be compact smooth oriented manifolds for which relative Pontriagin numbers can be defined, then

$$p_\omega(M_1^{4n} \coprod M_2^{4n}) = p_\omega(M_1^{4n}) - p_\omega(M_2^{4n})$$

and if $f : \partial M_1^{4n} \to \partial M_2^{4n}$ is a diffeomorphism then

$$p_\omega(M_1^{4n} \cup_f M_2^{4n}) = p_\omega(M_1^{4n}) - p_\omega(M_2^{4n})$$

1.3.- Brumfiel's invariant and cobordism

Theorem 1: [Brumfiel 1] a) *Let W^{4n} be a closed smooth spin manifold such that all its decomposable Pontriagin numbers are zero, ie., $p_{i_1\cdots i_r}(W^{4n}) = 0$, $r \geq 2$. Then the index of W^{4n}, $\sigma(W^{4n})$, is a multiple of $8t_n$.*

b) *Let $\Sigma^{4n-1} \in \Theta_{4n-1}$ be a homotopy sphere, then there is a compact smooth spin manifold M_0^{4n} with all its decomposable Pontriagin numbers zero and such that $\partial M_0^{4n} = \Sigma^{4n-1}$. Moreover, for any such M_0^{4n}, 8 divides the index $\sigma(M_0^{4n})$.*

As a consequence of this theorem Brumfiel defines a group homomorphism

$$f_R : \Theta_{4n-1} \to \mathbb{Z}_{t_n}$$

as follows. Let $\Sigma^{4n-1} \in \Theta_{4n-1}$. According to Theorem 1 *b)* above there is a spin manifold W^{4n} with all decomposable Pontriagin numbers zero and such that $\partial W^{4n} = \Sigma^{4n-1}$. Set now $f_R(\Sigma^{4n-1}) = \frac{1}{8}\sigma(W^{4n}) \in \mathbb{Z}_{t_n}$. This is well defined by a). Note that if Σ_0^{4n-1} is the Milnor sphere then $f_R(\Sigma_0^{4n-1}) = 1$. Therefore for n odd, $f_R|bP_{4n}$ is an isomorphism. For n even Brumfiel proves that $f_R|bP_{4n}$ has kernel 0 or $\mathbb{Z}_2$. But since the order of bP_{4n} is t_n for all n (see page 5) the kernel is always 0, not $\mathbb{Z}_2$.

Identify now bP_{4n} to $\mathbb{Z}_{t_n}$ (sending Σ_0^{4n-1} to 1). Then there is a split exact sequence

$$0 \to bP_{4n} \overset{f_R}{\underset{\to}{\leftarrow}} \Theta_{4n-1} \to \mathrm{Coker}(J_{4n-1}) \to 0$$

There is also a group homomorphism

$$f_R : \Theta_{8n+1} \to \mathbb{Z}_2$$

defined in [Brumfiel 2]. We quote also the following result [Brumfiel 3].

Proposition 2. *a) If $\Sigma_0^{8n+1} \in \Theta_{8n+1}$ is the Kervaire sphere then*

$$f_R(\Sigma_0^{8n+1}) = 1 \in \mathbb{Z}_2$$

b) Let W_1^{8n+2}, W_0^{8n+2} be compact smooth 1-connected spin manifolds with $\partial W_1^{8n+2} = S^{8n+1}$, $\partial W_0^{8n+2} = \Sigma^{8n+1} \in \Theta_{8n+1}$. If there exists a PL isomorphism of pairs

$$(\partial W_1^{8n+2}, S^{8n+1}) \to (W_0^{8n+2}, \Sigma^{8n+1})$$

then $f_R(\Sigma^{8n+1}) = 0$.
c) Identify bP_{8n+2} to $\mathbb{Z}_2$ (sending Σ_0^{8n+1} to 1) then $f_R : \Theta_{8n+1} \to \mathbb{Z}_2$ splits the exact sequence

$$0 \to bP_{8n+2} \overset{f_R}{\underset{\to}{\leftarrow}} \Theta_{8n+1} \to \mathrm{Coker}(J_{8n+1}) \to 0$$

Let Ω_*^{fr}, Ω_*^{Spin} and Ω_*^{SO} denote the framed, Spin and oriented cobordism rings ([Stong] and references quoted there), and let

$$\Omega_*^{fr} \xrightarrow{E_*} \Omega_*^{Spin}$$

$$\Omega_*^{Spin} \xrightarrow{F_*} \Omega_*^{SO}$$

denote the forgetful homomorphisms. It is well known that $F_* \circ G_* = 0$ in positive dimensions. The following result is taken from [Stong]; see also [ABP]:

THEOREM 3: (Anderson-Brown-Peterson) *a) The homomorphism* $E_*:\Omega_*^{fr}\to\Omega_*^{Spin}$ *has image* $\mathbb{Z}_2$ *in dimensions* 0, $8k+1$ *and* $8k+2$. *The image is zero otherwise.*
b) The homomorphism $F_*\otimes 1:\Omega_*^{Spin}\otimes\mathbb{Z}[\frac{1}{2}]\to\Omega_*^{SO}\otimes\mathbb{Z}[\frac{1}{2}]$ *is an isomorphism.*

1.4.- BREDON'S PAIRING

There is a map $\rho_{n,k}:\Theta_n\times\pi_{n+k}(S^n)\to\Theta_{n+k}$ defined by [Bredon]. The definition is as follows:
Let $(\Sigma^n,\alpha)\in\Theta_n\times\pi_{n+k}(S^n)$ with Σ^n represented by a diffeomorphism $h: S^{n-1}\to S^{n-1}$ and note that h can be assumed to be the identity outside an $(n-1)$-ball imbedded in S^{n-1}. Choose a framed submanifold $\langle M^k,F\rangle$ of S^{n+k} whose framed cobordism class corresponds to α. Take a closed tubular neighborhood T of M^k in S^{n+k} and let $f:M^k\times D^n\to T$ be a diffeomorphism corresponding to the framing F. Form the disjoint union $(S^{n+k}-\operatorname{int}T)+T$ and identify $x\in\partial T=\partial(S^{n+k}-\operatorname{int}T)$ to $f\circ(1\times h)\circ f^{-1}(x)$ where $1\times h: M^k\times S^{n-1}\to M^k\times S^{n-1}$. The resulting manifold is clearly a homotopy $(n+k)$-sphere and determines an element $\rho_{n,k}(\Sigma^n,\alpha)\in\Theta_{n+k}$. It is proved in [Bredon] that $\rho_{n,k}$ is well defined, as well as the following:
PROPOSITION 4: (Bredon) *a) If* $k-1<n$ *then* $\rho_{n,k}$ *is bilinear.*
b) Let $p':\Theta_n\to\operatorname{Coker}J_n$ *be the map induced by Thom Pontriagin construction (see [Kervaire-Milnor]) and for* $k-1<n$ *let* $c':\operatorname{Coker}J_n\times\pi_{n+k}(S^n)\to\operatorname{Coker}J_{n+k}$ *be induced by composition, then the following diagram commutes:*

$$\begin{array}{ccc}\Theta_n\times\pi_{n+k}(S^n) & \xrightarrow{\rho_{n,k}} & \Theta_{n+k}\\ p'\times 1\downarrow & & \downarrow p'\\ c':\operatorname{Coker}J_n\times\pi_{n+k}(S^n) & \xrightarrow{c'} & \operatorname{Coker}J_{n+k}\end{array}$$

The condition $k-1<n$ on part *b)* is necessary to define c' and to prove that the diagram commutes.

2. The invariant g_R

2.1.- Definition of g_R and byproducts

First we state and prove the following consequence of Theorem 3.

Proposition 5: *Let W^{4n} be a closed oriented smooth manifold, then there is some closed spin manifold X^{4n} and an integer $c \geq 0$ such that $2^c[W^{4n}] = [X^{4n}]$ in Ω_*^{SO}.*

Proof: By Theorem 3, $F_* \otimes 1 : \Omega_*^{Spin} \otimes \mathbb{Z}[\frac{1}{2}] \to \Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism. Therefore there are spin manifolds X_i^{4n} and numbers $a_i = m_i/2^{b_i} \in \mathbb{Z}[\frac{1}{2}]$, $i = 1, \ldots, s$, such that $[W^{4n}] \otimes 1 = \sum_{i=1}^{s} [X_i^{4n}] \otimes a_i$ in $\Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$. If b is a large enough integer then $2^i a_i$ is an integer for all i. Therefore

$$\begin{aligned} 2^b[W^{4n}] \otimes 1 &= \textstyle\sum_{i=1}^{s} [X_i^{4n}] \otimes 2^b a_i \\ &= [\textstyle\sum_{i=1}^{s} 2^b a_i X_i^{4n}] \otimes 1 \end{aligned}$$

in $\Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$. Let Y^{4n} be the spin manifold (disjoint union) $\sum_{i=1}^{s} 2^b a_i X_i^{4n}$, then $(2^b[W^{4n}] - [Y^{4n}]) \otimes 1 = 0$ in $\Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}]$ hence $2^b[W^{4n}] - [Y^{4n}]$ is a 2-torsion element in Ω_*^{SO}, that is, there exists some integer d such that $2^{b+d}[W^{4n}] - 2^d[Y^{4n}] = 0$ in Ω_*^{SO}. Take $X^{4n} = 2^d Y^{4n}$ and $c = b + d$ Q.E.D.

Let t_n and t'_n be as in pages 2 and 3.

Theorem 6: *Let W^{4n} be a closed oriented smooth manifold with all decomposable Pontriagin numbers zero. Then t'_n divides the index $\sigma(W^{4n})$.*

Proof: By previous proposition there is a spin manifold X^{4n} such that $2^c W^{4n}$ is cobordant to X^{4n}. Then $\sigma(X^{4n} = 2^c \sigma(W^{4n}$ and $p_\omega(X^{4n}) = 2^c p_\omega(W^{4n}) = 0$. By Theorem 1, $8t_n$ divides $\sigma(X^{4n})$ and therefore t'_n divides $\sigma(W^{4n})$ Q.E.D.

Proposition 7: *Let W^{4n} be a closed oriented smooth manifold with all decomposable Pontriagin numbers zero, then $p_n(W^{4n})$ is a multiple of $r_n =$* odd part of $(2n-1)! \cdot \text{denom}\,(B_n/4n)$.

Proof: By the Hirzebruch index theorem (see [Hirzebruch], pages 12 and

86)

$$\begin{aligned}
\sigma(W^{4n}) &= \frac{2^{2n}(2^{2n-1}-1)}{(2n)!}B_n \cdot p_n \\
&= \frac{4n\,2^{2n}(2^{2n-1}-1)}{(2n)!}\frac{B_n}{4n} \cdot p_n \\
&= \frac{2^{2n+1}(2^{2n-1}-1)}{(2n-1)!}\frac{\mathrm{num}\,(B_n/4n)}{\mathrm{denom}\,(B_n/4n)} \cdot p_n \\
&= \frac{2^{2n+1}}{(2n-1)!}\frac{2^c t'_n}{\mathrm{denom}\,(B_n/4n)} \cdot p_n \\
&= \frac{2^{2n+c+1} \cdot t'_n}{2^d \cdot r_n} \cdot p_n \\
&= 2^{2n+c-d+1}\frac{t'_n}{r_n} \cdot p_n
\end{aligned}$$

From this one obtains

$$2^{2n+c+d+1} \cdot p_n = \frac{\sigma(W^{4n})}{t'_n} \cdot r_n$$

By previous proposition $\sigma(W^{4n})/t'_n$ is an integer, and since r_n is odd it follows that $r_n | p_n$ Q.E.D.

REMARK: $r_n \geq (2n-1) \cdot (2n-3) \cdots 3 \cdot 1$.

COROLLARY 8: *Let $\partial W_1^{4n} = \Sigma^{4n-1} \in \Theta_{4n-1}$, where W_1^{4n} is compact, oriented and has all decomposable Pontriagin numbers equal to zero, then*

$$8 f_R(\Sigma^{4n-1}) \equiv \sigma(W_1^{4n}) \bmod t'_n$$

PROOF: Let $\partial W_0^{4n} = \Sigma^{4n-1}$ where W_0^{4n} is spin and has all decomposable Pontriagin numbers 0 (see Theorem 1). Form the manifold $W^{4n} = W_0^{4n} \cup_f (-W_1^{4n})$ where $f : \partial W_0^{4n} \to \partial W_1^{4n}$ is an orientation preserving diffeomorphism. Then the decomposable Pontriagin numbers of W^{4n} are $p_\omega(W^{4n}) = p_\omega(W_0^{4n} \cup_f (-W_1^{4n})) = p_\omega(W_0^{4n}) - p_\omega(W_1^{4n}) = 0 - 0 = 0$ and the index is

$$\begin{aligned}
\sigma(W^{4n}) &= \sigma(W_0^{4n}) - \sigma(W_1^{4n}) \\
&= 8 f_R(\Sigma^{4n-1}) - \sigma(W_1^{4n}) \in \mathbb{Z}_{t'_n}
\end{aligned}$$

By Theorem 6, t'_n divides $\sigma(W^{4n})$, therefore $8f_R(\Sigma^{4n-1}) \equiv \sigma(W_1^{4n}) \bmod t'_n$ Q.E.D.

It is always possible to define the following group homomorphism $g_R : \Theta_{4n-1} \to \mathbb{Z}_{t'_n}$: For $\Sigma^{4n-1} \in \Theta_{4n-1}$ define $g_R(\Sigma^{4n-1})$ as $f_R(\Sigma^{4n-1})$ mod t'_n, that is, as the image of Σ^{4n-1} under the composition

$$\Theta_{4n-1} \xrightarrow{f_R} \mathbb{Z}_{t_n} \to \mathbb{Z}_{t'_n}$$

where the last arrow is the canonical projection.

Although g_R is a coarser invariant than f_R its significance stems from the fact that, according to Corollary 8, $g_R(\Sigma^{4n-1})$ can be computed from any compact oriented (not necessarily spin) W_1^{4n} with boundary Σ^{4n-1}. For any such W_1^{4n} one has

$$g_R(\Sigma^{4n-1}) = (1/8)\sigma(W_1^{4n}) \in \mathbb{Z}_{t'_n}$$

Note that $(1/8) \in \mathbb{Z}_{t'_n}$ because t'_n is odd. Also, if Σ_0^{4n-1} is the Milnor sphere then $g_R(\Sigma_0^{4n-1}) = 1$.

It is possible to extend further the definition of g_R as follows. Let M^{4n-1} be a closed oriented smooth manifold satisfying the following conditions:

a) $\partial W^{4n} = M^{4n-1}$ for some compact, oriented and smooth manifold W^{4n}.

b) Relative decomposable Pontriagin numbers of W^{4n} can be defined (see p.17) and are all zero.

Define then

$$g_R(M^{4n-1}) = \frac{1}{8}\, s(W^{4n}) \text{ mod } t'_n$$

Proposition 9: *The value $g_R(M^{4n-1}) \in \mathbb{Z}_{t'_n}$ depends only on the oriented diffeomorphism class of M^{4n-1}.*

Proof: Let W_0^{4n}, W_1^{4n} be $4n$-manifolds with $\partial W_i^{4n} = M^{4n-1}$, $i = 0, 1$ and consider any orientation preserving diffeomorphism $f : M^{4n-1} = \partial W_1^{4n} \to M^{4n-1} = \partial W_0^{4n}$.

Form the smooth oriented manifold $W^{4n} = W_1^{4n} \bigcup_f (-W_0^{4n})$ then W^{4n} is closed, oriented and has decomposable Pontriagin numbers $p_\omega(W^{4n}) = p_\omega(W_1^{4n}) - p_\omega(W_0^{4n}) = 0 - 0 = 0$.

Theorem 6 states that $\sigma(W^{4n})$ is divisible by t'_n and since $\sigma(W_0^{4n}) - \sigma(W_1^{4n}) = \sigma(W^{4n})$ and t'_n is odd, it follows that $(1/8)\sigma(W_0^{4n}) \equiv \sigma(W_1^{4n}) \bmod t'_n$, Q.E.D.

PROPOSITION 10: *If* M_0^{4n-1}, M_1^{4n-1} *are* bE_{4n} *manifolds then*

$$g_R(M_1^{4n-1}\#(-M_0^{4n-1})) = g_R(M_1^{4n-1}) - g_R(M_0^{4n-1})$$

PROOF: Let W_0^{4n}, W_1^{4n} be E_{4m} manifolds with $\partial W_0^{4n} = M_0^{4n-1}$, $\partial W_1^{4n} = M_1^{4n-1}$. Then $W_1^{4n}\coprod(-W_0^{4n})$ has boundary $\partial(W_1^{4n}\coprod(-W_0^{4n})) = M_1^{4n-1}\#(-M_0^{4n-1}$ and therefore

$$\begin{aligned} g_R(M_1^{4n-1}\#(-M_0^{4n-1})) &= (1/8)\sigma(W_1^{4n}\textstyle\coprod(-W_0^{4n}) \\ &= (1/8)\sigma(W_1^{4n}) - (1/8)\sigma(W_0^{4n}) \\ &= g_R(M_1^{4n-1}) - g_R(M_0^{4n-1}) \end{aligned}$$

Q.E.D.

3. Inertia and concordance inertia groups

3.1.- PD maps .

We discuss here PD maps. For the definition of rectilinear cell complex, smooth triangulation of a smooth manifold, etc. see [Munkres 1] Chapter 2.

Let K be a rectilinear cell complex, M, N smooth manifolds and $\alpha : K \to M$ a smooth triangulation. A map $f : M \to N$ is piecewise differentiable, abbreviated PD on the triangulation α if $f \circ \alpha : K \to N$ is smooth on each simplex of some subdivision K' of K. When there is no danger of confusion, the triangulation α on which f is PD will not be mentioned explicitly.

The map $f : M \to N$ is a PD isomorphism if for some subdivision K' of K the map $f \circ \alpha : K' \to N$ is a smooth triangulation of N

Let $x_0 \in M^n$ and let $f : M^n \to N^n$ be a homeomorphism with the property of being a diffeomorphism on $M^n - \{x_0\}$. If $\alpha : K \to M^n$ is a triangulation of M^n let K' be a subdivision of K such that $\alpha^{-1}(x_0)$ is a vertex of K' (such triangulation always exists) then $f \circ \alpha : K' \to N^n$ is a smooth triangulation of N^n and therefore f is a PD-isomorphism on any α.

Let $\eta : K \to L$ be a PL isomorphism and $\beta : L \to N^n$ a smooth triangulation of N^n. There is a subdivision K' of K and a subdivision L' of L such that η carries each simplex s of K' onto some simplex t of L' via a linear isomorphism. Since $\beta : L \to N^n$ is a smooth triangulation it follows that $\beta \circ \eta$ is a diffeomorphism on each simplex σ of K'. Therefore $\beta \circ \eta$ is a diffeomorphism on each simplex s of K' ([Munkres 1], 8.4).

Consider now triangulations

$$\alpha : K \to M^n, \quad \beta : L \to N^n$$

Suppose that $f : M^n \to N^n$ is a map such that

$$\eta = \beta^{-1} \circ f \circ \alpha : K \to L$$

is a PL isomorphism. By previous discussion it follows that

$$\beta \circ \eta = \beta \circ \beta^{-1} \circ f \circ \alpha = f \circ \alpha : K' \to N^n$$

is a smooth triangulation of N^n. Therefore f is a PD-isomorphism on α.

Let $\alpha : K \to M^n$, $\beta : L \to N^n$ be smooth triangulations. Any map $f : M^n \to N^n$ induces a map $f_\alpha^\beta = \beta^{-1} \circ f \circ \alpha : K \to L$ between triangulations. If f is a PD isomorphism between the manifolds (in particular if f

is a diffeomorphism) then f_α^β is a PD-isomorphism between the complexes; generally speaking f_α^β is not a PL map. However with arbitrarily small perturbations of β it is possible to obtain a PL isomorphism, and this can be done without perturbing β on subcomplexes of K where β is already PL. More precisely, assume that f_α^β is a PL isomorphism on some closed subcomplex $A \subseteq K$ and let $B = f_\alpha^\beta(A)$. Then, according [Munkres 1] 10.13, there is for any $\delta > 0$ a δ-approximation $\gamma : L \to N^n$ to β, which restricted to B equals β, $\gamma|B = \beta$ and such that f_α^γ is a PL isomorphism. As a consequence, if M^n and N^n are PD-isomorphic then they are PL isomorphic

3.2.- Inertia Groups.

For the remaining of this paper the manifolds M^n under study will be assumed to have dimension $n \geq 7$. This requirement is necessary in order to have f_R to work and to be able to approximate certain maps by isotopies.

Consider the following particular presentation of the connected sum of a closed connected smooth manifold M^n with a homotopy n-sphere Σ^n. Let

$$g : S^{n-1} \to S^{n-1}$$

be a diffeomorphism corresponding to Σ^n and let

$$\varphi : D^n \to M^n$$

be an orientation preserving embedding with $\varphi(0) = x_0$.

To define the connected sum $M^n \# \Sigma^n$ define an atlas on M^n as follows. On $M^n - \{x_0\}$ take the atlas induced from the original smoothing on M^n. At the point x_0 consider the chart

$$\varphi \circ Cg^{-1} \circ \varphi^{-1} : \varphi(D^n) \to D^n$$

Here

$$Cg^{-1} : D^n \to D^n$$

is the cone extension of g^{-1} so that Cg^{-1} is a homeomorphism, smooth except at $0 \in D^n$. This chart $\varphi \circ Cg^{-1} \circ \varphi^{-1}$ is compatible with the charts on $M^n - \{x_0\}$. So, we have an atlas on M^n and therefore a smoothing. The topological manifold M^n with this smoothing is the connected sum of M^n and Σ^n, denoted $M^n \# \Sigma^n$. So M^n and $M^n \# \Sigma^n$ are the same topological space.

The identity map $1_{M^n} : M^n \to M^n \# \Sigma^n$ is smooth except at x_0 and is therefore a PD-isomorphism in any triangulation α, and $1_{M^n} \circ \alpha : K' \to$

$M^n\#\Sigma^n$ is a smooth triangulation for some subdivision K' of K. If M^n is diffeomorphic to N^n (abbreviated $M^n \cong N^n$) then $M^n\#\Sigma^n \cong N^n\#\Sigma^n$ and $M^n\#(\Sigma_1^n\#\Sigma_2^n) \cong (M^n\#\Sigma_1^n)\#\Sigma_2^n$.

The inertia group of M^n, $I(M^n)$, consists of those homotopy spheres such that $M^n \cong M^n\#\Sigma^n$.

$I(M^n)$ is certainly a group. For let $\Sigma^n \in I(M^n)$, then

$$M^n\#(-\Sigma^n) \cong (M^n\#\Sigma^n)\#(-\Sigma^n) \cong M^n\#(\Sigma^n\#(-\Sigma^n)) \cong M^n\#S^n \cong M^n$$

so that if $\Sigma^n \in I(M^n)$, then $-\Sigma^n \in I(M^n)$

Also, if Σ_1^n, $\Sigma_2^n \in I(M^n)$ then

$$M^n\#(\Sigma_1^n\#\Sigma_2^n) \cong (M^n\#\Sigma_1^n)\#\Sigma_2^n \cong M^n\#\Sigma_2^n \cong M^n$$

so that $\Sigma_1^n\#\Sigma_2^n \in I(M^n)$. Hence $I(M^n)$ is a group.

Let $\Sigma^n \in I(M^n)$ so that there is an orientation preserving diffeomorphism $\bar{f} : M^n \to M^n\#\Sigma^n$. In what follows we use the notation of the first two paragraphs of this chapter. We have that

$$\bar{f} \circ \varphi : D^n \to M^n\#\Sigma^n$$

and

$$\bar{f} \circ Cg : D^n \to M^n\#\Sigma^n$$

are orientation preserving embeddings. By the Palais-Cerf Lemma ([Milnor 3]) these embeddings are isotopic. By the isotopy extension theorem (ibid) there is an ambient isotopy

$$h_t : M^n\#\Sigma^n \to M^n\#\Sigma^n$$

such that h_0 =identity and

$$h_1 \circ \bar{f} = \varphi \circ Cg : D^n \to M^n\#\Sigma^n$$

Therefore $\bar{\bar{f}} = h_1 \circ \bar{f} : M^n \to M^n\#\Sigma^n$ is a diffeomorphism such that $\bar{\bar{f}}(\varphi(D^n)) = \varphi(D^n)$. Also, $\bar{\bar{f}}$ and $\bar{f}$ are in the same isotopy class. If $\psi : D^n \to M^n$ is another embedding with $\psi(D^n)\cap\varphi(D^n) = \emptyset$ we can use the Palais-Cerf Lemma again and conclude that $\bar{\bar{f}}$ can be chosen so that $\bar{\bar{f}}$ =identity on some neighborhood of $\psi(D^n)$. We have proved

LEMMA 11: *If $\Sigma^n \in I(M^n)$, there is an orientation preserving diffeomorphism $\bar{\bar{f}} : M^n \to M^n \# \Sigma^n$ such that $\bar{\bar{f}}(\varphi(D^n)) = \varphi(D^n)$ and $\bar{\bar{f}}$ = identity on some neighborhood of $\psi(D^n)$.*

Let M_0^n be $M_0^n - \mathrm{int}(\varphi(D^n))$. Then M_0^n is a smooth submanifold with boundary of M and of $M^n \# \Sigma^n$. Suppose now that $\Sigma^n \in I(M^n)$ and let $\bar{\bar{f}} : M^n \to M^n \# \Sigma^n$ be as in Lemma 11. From the definition of the smoothing on $M^n \# \Sigma^n$ we have that the following composition is a diffeomorphism:

$$(*) \qquad D^n \xrightarrow{\varphi} \varphi(D^n) \xrightarrow{\bar{\bar{f}}|\varphi(D^n)} \varphi(D^n) \xrightarrow{\varphi^{-1}} D^n \xrightarrow{Cg^{-1}} D^n$$

Recall now that two orientation preserving diffeomorphisms

$$h, \bar{h} : S^{n-1} \to S^{n-1}$$

are isotopic if and only if

$$\bar{h}^{-1} \circ h : S^{n-1} \to S^{n-1}$$

extends to a diffeomorphism of D^n ([Milnor 3]). Therefore, since

$$g^{-1} \circ (\varphi^{-1} \circ \bar{\bar{f}} \circ \varphi)|S^{n-1}$$

extends by (*) to D^n, we conclude that g and $\varphi^{-1} \circ \bar{\bar{f}} \circ \varphi|S^{n-1}$ are isotopic. Let $g_t : S^{n-1} \to S^{n-1}$ be an isotopy between $g_0 = g$ and $g_1 = \varphi^{-1} \circ \bar{\bar{f}} \circ \varphi$. Then

$$\varphi^{-1} \circ g_t \circ \varphi : \partial M_0^n \to \partial M_0^n$$

is an isotopy between $\varphi^{-1} \circ g \circ \varphi : \partial M_0^n \to \partial M_0^n$ and $\varphi \circ \varphi^{-1} \circ \bar{\bar{f}} \circ \varphi \circ \varphi^{-1} = \bar{\bar{f}}|\partial M_0^n : \partial M_0^n \to \partial M_0^n$. Since $\bar{\bar{f}}|\partial M_0^n$ extends (by $\bar{\bar{f}}$) to M_0^n it follows from [Palais] that

$$\varphi \circ g \circ \varphi^{-1} : \partial M_0^n \to \partial M_0^n$$

also extends to M_0^n. Note that we can assume the extension to be the identity on some neighborhood of $\psi(D^n)$. We have proved half of the following

PROPOSITION 12: *$\Sigma^n \in I(M^n)$ if and only if the diffeomorphism $\varphi \circ g \circ \varphi^{-1} : \partial M_0^n \to \partial M_0^n$ extends to a diffeomorphism f_0 of M_0^n such that f_0 = identity on some neighborhood of $\psi(D^n)$.*

PROOF: It remains to consider the "if" part. Suppose that $\varphi \circ g \circ \varphi^{-1} : \partial M_0^n \to \partial M_0^n$ extends to a diffeomorphism $h : M_0^n \to M_0^n$. Define $f : M^n \to M^n \# \Sigma^n$ as follows: Let $f|M_0^n = h$ and let $f|\varphi(D^n) = h$ be

$$\varphi \circ Cg \circ \varphi^{-1} : \varphi(D^n) \subseteq M^n \to \varphi(D^n) \subseteq M^n \# \Sigma^n$$

then, checking with the chart $\varphi \circ Cg^{-1} \circ \varphi^{-1}$ at $x_0 \in M^n \# \Sigma^n$, we have that $f : M^n \to M^n \# \Sigma^n$ is a diffeomorphism. QED

COROLLARY 12': *$\Sigma^n \in I(M^n)$ if and only if there is a diffeomorphism $f : M^n \to M^n \# \Sigma^n$ such that*

$$f|\varphi(D^n) = \varphi \circ Cg \circ \varphi^{-1}$$

and $f =$ identity on some neighborhood of $\psi(D^n)$.

PROOF: Let $f_0 : M_0^n \to M_0^n$ be as in Lemma 12, then f_0 extends to $f : M^n \to M^n \# \Sigma^n$ if we let $f|\varphi(D^n) = \varphi \circ Cg \circ \varphi^{-1}$ (check with the appropriate chart). QED

A diffeomorphism $f : M^n \to M^n \# \Sigma^n$ satisfying the conditions of Corollary 12' will be called an i-diffeomorphism. We have proved

THEOREM 13: *In any isotopy class of orientation preserving diffeomorphisms $f : M^n \to M^n \# \Sigma^n$ there is an i-diffeomorphism.* Q.E.D.

The following proposition provides an upper bound for the inertia groups of some manifolds.

PROPOSITION 14: *Let M^{4n-1} be a closed, smooth, oriented and connected manifold which bounds some manifold W_0^{4n} satisfying b) of page 15. Then*

$$I(M^{4n-1}) \subseteq \ker g_R$$

PROOF: If $M^{4n-1} \cong M^{4n-1} \# \Sigma^{4n-1}$, we have that

$$g_R(M^{4n-1} \# \Sigma^{4n-1}) = g_R(M^{4n-1}) - g_R(\Sigma^{4n-1})$$

Therefore

$$g_R(\Sigma^{4n-1}) \equiv 0 \mod t'_n$$

Q.E.D.

3.3.- CONCORDANCE INERTIA GROUPS

The concordance inertia group of M^n, $I_c(M^n) \subseteq \Theta_n$, consists of those homotopy spheres Σ^n such that for some triangulation $\alpha : K \to M^n$, M^n and $M^n \# \Sigma^n$ are diffeomorphic via a diffeomorphism $\bar{f} : M^n \to M^n \# \Sigma^n$ which is PD concordant to the identity $1 : M^n \to M^n \# \Sigma^n$.

REMARK: Let PL/O be the space defined in [Milnor 4] and let $M^n \to S^n$ be a degree one map. Consider a closed connected smooth manifold M^n.

There is a one-to-one correspondence between the set of concordance classes of smoothings of M^n and the set of homotopy classes $[M^n, PL/O]$ (ibid), in particular, $\pi_n(PL/O) \cong \Theta_n$ ([Hirsch]). Then, the concordance inertia group $I_c(M^n)$ fits into an exact sequence of pointed sets

$$0 \to I_c(M^n) \to \Theta_n \cong [S^n, PL/O] \to [M^n, PL/O]$$

Let the diffeomorphism $\bar{f}$ above be smoothly isotopic to a diffeomorphism $f : M^n \to M^n \# \Sigma^n$. Then $\bar{f} \sim_c f$, and since $\bar{f} \sim_c 1$ it follows that $f \sim_c 1$.

From Theorem 13, we know that there is an i-diffeomorphism $f : M^n \to M^n \# \Sigma^n$ smoothly isotopic to f, hence:

Proposition 15: $\Sigma^n \in I_c(M^n)$ *if and only if there is an i-diffeomorphism $f : M^n \to M^n \# \Sigma^n$ which is PD concordant to the identity map.*

Proof: The 'only if' part is just the previous discussion. The 'if' part is trivial. Q.E.D.

Consider now $M_0^n = M^n - \operatorname{int} \varphi(D^n)$, $\partial M_0^n = \varphi(S^{n-1})$, $\Sigma^n \in \Theta_n$ and $g : S^{n-1} \to S^{n-1}$ a corresponding diffeomorphism.

Proposition 16: *If $\varphi \circ g \circ \varphi^{-1}$ extends to a diffeomorphism $f_0 : M_0^n \to M_0^n$ such that f_0 is PD isotopic to the identity $1_{M_0^n} : M_0^n \to M_0^n$, then $\Sigma^n \in I_c(M^n)$.*

Proof: Let $F : M_0^n \times I \to M_0^n \times I$ be a PD isotopy from f_0 to $1_{M_0^n}$, on the triangulation $\alpha_0 : K_0 \to M_0^n$. First extend α_0 to a triangulation

$$\alpha : K_0 \cup \operatorname{Cone}(\partial K_0) \to M^n$$

with α defined by the following condition:

$$\alpha | K_0 = \alpha_0 \qquad \alpha[(t, x)] = \varphi t \varphi^{-1} \alpha(x)$$

where $0 \leq t \leq 1$, $x \in \partial K_0$ and $[(t, x)] \in \operatorname{Cone}(\partial K_0)$. For each t, let $F_t : M_0^n \to M_0^n$ be defined by

$$F(x, t) = (F_t(x), t)$$

then $h_t : \varphi^{-1} \circ F_t \circ \varphi | S^{n-1} : S^{n-1} \to S^{n-1}$ extends by the cone extension to a map

$$Ch_t : D^n \to D^n$$

Let $\bar{F}_t : M^n \times \{t\} \to M \times \{t\}$ be the extension of F_t defined by the following condition

$$\bar{F}_t | \varphi(D^n) = \varphi \circ Ch_t \circ \varphi^{-1}$$

Then $\bar{F}_0$ is a diffeomorphism $M^n \to M^n \# \Sigma^n$ (in fact, $\bar{F}_0|\varphi(D^n) = \varphi \circ Cg \circ \varphi^{-1}$), and $\bar{F}_1$ =identity. Then the map $\bar{F} : M^n \times I \to M^n \times I$ defined by

$$\bar{F}(x,t) = (\bar{F}_t(x), t)$$

is a PD isotopy (on α) from the diffeomorphism $\bar{F}_0 : M^n \to M^n \# \Sigma^n$ to $1_{M^n} : M^n \to M^n \# \Sigma^n$. Therefore $\Sigma^n \in I_c(M^n)$ Q.E.D.

3.4.- Upper bound for $I_c(M^{4n-1})$

Continue with the notations of 3.2, i.e., $\varphi : D^n \to D^n$ is an embedding etc. Define

$$\Delta_+^{n+1} = \{(x_1, \ldots, x_{n+1}) \in D^{n+1} \mid x_{n+1} \geq 0\}$$
$$\Delta_-^{n+1} = \{(x_1, \ldots, x_{n+1}) \in D^{n+1} \mid x_{n+1} \leq 0\}$$

then $\Delta_+^{n+1} \cup \Delta_-^{n+1} = D^{n+1}$ and $D_+^n \subseteq \Delta_+^{n+1}$, $D_-^n \subseteq \Delta_-^{n+1}$. Consider now smooth embeddings

$$e_+ : \Delta_+^{n+1} \to M^n \times I \qquad e_+ : \Delta_+^{n+1} \to M^n \times I$$

defined as follows

$$e_+(x) = e_+(x_1, \ldots, x_{n+1}) = (\varphi((1/6)x_1, \ldots, (1/6)x_n), (1/6)x_{n+1})$$

$$e_-(x) = e_-(x_1, \ldots, x_{n+1}) = (\varphi((1/6)x_1, \ldots, (1/6)x_n), 1 + ((1/6)x_{n+1}))$$

Note that if $x_{n+1} = 0$ then

$$e_+(x) = (\varphi((1/6)x), 0)$$

$$e_-(x) = (\varphi((1/6)x), 1)$$

If on $M^n \times I$ $(x, 0)$ is identified to $(x, 1)$, the quotient space is a smooth manifold naturally isomorphic to $M^n \times S^1$. So this quotient of $M^n \times I$ will be denoted by $M^n \times S^1$.

If $(x, t) \in M^n \times I$, denote by $[x, t]$ its image on $M^n \times S^1$ under the quotient map. Consider

$$V^{n+1} = M^n \times I - \mathrm{int}(e_+(\Delta_+^{n+1}) \cup e_-(\Delta_-^{n+1}))$$

If on V^{n+1} $(x, 0) \in V^{n+1} \subseteq M^n \times I$ is identified to $(x, 1) \in V^{n+1} \subseteq M^n \times I$, the resulting smooth manifold will be the same as $M^n \times S^1-$open disk and this quotient of V^{n+1} will be denoted by $(M^n \times S^1)_0$. If $(x,t) \in V^{n+1}$ let $[x,t]$ denote its image on $(M^n \times S^1)_0$ under the quotient map. Similarly, $[A]$ is the image of the subset $A \subseteq V^{n+1}$. Finally $(M^n \times S^1)_0 \subseteq M^n \times S^1$ and $\partial((M^n \times S^1)_0) = S^n$.

Since $(M^n \times S^1)_0 = M^n \times S^1 -$open disk the index of $(M^n \times S^1)_0$ is 0, for

$$s((M^n \times S^1)_0) = s(M^n \times S^1) = s(M^n)s(S^1) = 0$$

and for Pontriagin numbers

$$p_\omega((M^n \times S^1)_0) = p_\omega(M^n \times S^1) = p_\omega(M^n \times \partial D^2) = p_\omega(\partial(M^n \times D^2)) = 0$$

Now let $\Sigma^n \in I_c(M^n)$ and $F : M^n \times I \to (M^n \# \Sigma^n) \times I$ a PD concordance between the i-diffeomorphism $f : M^n \to M^n \# \Sigma^n$ and the identity $1 : M^n \to M^n \# \Sigma^n$. It may be assumed that

$$F(x,t) = (f(x),t) \text{ for} 0 \leq t \leq (1/3)$$

and that

$$F(x,t) = (x,t) \text{ for} (2/3) \leq t \leq 1$$

Let X_2^{n+1} be the smooth manifold obtained from

$$\begin{aligned} V^{n+1} &= M^n \times I - \operatorname{int}(e_+(\Delta_+^{n+1}) \cup e_-(\Delta_-^{n+1})) \\ &= (M^n \# \Sigma^n) \times I - \operatorname{int}(e_+(\Delta_+^{n+1}) \cup e_-(\Delta_-^{n+1})) \end{aligned}$$

identifying $(x,1) \in V^{n+1}$ to $(f(x),0) \in V^{n+1}$. This identification is well defined since

$$f|\varphi(D^n) = \varphi \circ Cg \circ \varphi^{-1}$$

∂X_2^{n+1} is the manifold obtained from the disjoint union

$$e_+(D_+^n) + e_-(D_-^n)$$

identifying

$$(z,1) = (\varphi((1/6)x),1) \in e_+(S^{n-1})$$

to

$$\begin{aligned} (f(z),0) &= (f(\varphi((1/6)x),0) \\ &= (\varphi \circ \mathrm{C}g \circ \varphi^{-1} \circ \varphi((1/6)x),0)) \\ &= (\varphi((1/6)g(x)),0) \in e_-(S^{n-1}) \end{aligned}$$

for each $x \in S^{n-1}$.

This boundary is therefore diffeomorphic to the homotopy sphere Σ^n obtained from the disjoint union $D^n_+ + D^n_-$ identifying $x \in D^n_+$ to $g(x) \in D^n_-$.

An explicit diffeomorphism $h : \Sigma^n \to \partial X_2^{n+1}$ is defined by

$$\begin{array}{rcll} (h|D^n_+)(x) & = & e_+(x), & x \in D^n_+ \\ (h|D^n_-)(x) & = & e_-(g(x)), & x \in D^n_- \end{array}$$

The PD concordance $F : M^n \times I \to (M^n \# \Sigma^n) \times I$ restricts to a map $V^{n+1} \to V^{n+1}$ which induces a PD isomorphism of pairs

$$H : ((M^n \times S^1)_0, \partial((M^n \times S^1)_0)) \to (X_2^{n+1}, \partial X_2^{n+1})$$

Therefore X_2^{n+1} and $(M^n \times S^1)_0$ are PL isomorphic. It follows that if $n \equiv -1 \mathrm{mod} 4$, the index satisfies

$$s(X_2^{n+1}) = s((M^n \times S^1)_0) = 0$$

and

$$p_\omega(X_2^{n+1}) = p_\omega((M^n \times S^1)_0) = 0$$

for all ω.

THEOREM 17: *If M^{4n-1} is closed smooth and orientable manifold then*

$$I_c(M^{4n-1}) \subseteq \ker g_R$$

If M^{4n-1} is also spin then $I_c(M^{4n-1}) \subseteq \ker f_R$.

PROOF: Let $\Sigma^{4n-1} \in I_c(M^{4n-1})$ and let X_2^{4n} be the manifold constructed above, with $\partial X_2^{4n} \cong \Sigma^{4n-1}$ then X_2^{4n} has all decomposable Pontriagin numbers and index equal to zero, hence $g_R(\Sigma^{4n-1}) = s(X_2^{4n}) = 0 \in \mathbb{Z}_{t'_n}$.

If M^{4n-1} is spin, so are $M^{4n-1} \times S^1$ and $(M^{4n-1} \times S^1)_0$. Since X_2^{4n} is PL isomorphic to $(M^{4n-1} \times S^1)_0$,

$$w_2(X_2^{4n}) = w_2((M^{4n-1} \times S^1)_0) = 0$$

hence X_2^{4n} is spin, and $f_R(\Sigma^{4n-1}) = s(X_2^{4n}) = 0 \in \mathbb{Z}_{t'_n}$. Q.E.D.

In particular, the Milnor sphere Σ_0^{4n-1} is not in $I_c(M^{4n-1})$ for any orientable manifold M^{4n-1}. Moreover since $\ker g_R \cap bP_{4n}$ consists of elements of order 2, it has at most $2^{2n-2} \cdot a_n$ elements, $I_c(M^{4n-1}) \cap bP_{4n}$ has at most $2^{2n-2} \cdot a_n$

elements which are all of order 2. If M^{4n-1} is spin, since ker $f_R \cap bP_{4n} = 0$, $I_c(M^{4n-1}) \cap bP_{4n} = 0$ for any spin manifold M^{4n-1}.

3.5.- UPPER BOUND FOR $I_c(M^{8n+1})$

In order to prove results in dimensions $\equiv 1(\text{mod}\, 8)$ we need the next PROPOSITION. Assume the notation of 3.4 so that $\varphi, \psi : D^n \to M^n$, $\varphi(D^n) \cap \psi(D^n) = \emptyset$, etc.

PROPOSITION 18: *Let $\Sigma^n \in I_c(M^n)$. Then there is an i-diffeomorphism*

$$f : M^n \to M^n \# \Sigma^n$$

a PD concordance

$$F : M^n \times I \to (M^n \# \Sigma^n) \times I$$

and an open set

$$U \subseteq \psi(D^n) \qquad y_0 = \psi(0)$$

such that

$$F|U \times I \cup M^n \times [(2/3), 1] = \text{identity}$$

and $F(x,t) = (f(x), t)$ for $0 \leq t \leq 1/3$.

PROOF: The proof will be postponed until Chapter 6.

It can be assumed above that U is an n-disk in M^n. Let

$$\begin{array}{rcl} A' & = & U \times I \cup M^n \times ((2/3), 1) \subseteq V^{n+1} \\ B' & = & \{y_0\} \times I \cup M^n \times \{11/12\} \subseteq A' \end{array}$$

Since $f|\psi(D^n)$ is the identity, and $U \subseteq \psi(D^n)$, the image $[A']$ of A' under the identification map $V^{n+1} \to (M^n \times I)_0$ is equal to its image under the identification map $V^{n+1} \to X_2^{n+1}$. Call A this common image. Then A is naturally isomorphic to

$$U \times S^1 \cup [M^n \times (2/3, 1)]$$

and is an open submanifold of both $(M^n \times I)_0$ and X_2^{n+1}.

Since $B' \subseteq A'$, if we call B the image $[B]$ of B' under either identification map, we have that $B \subseteq A$. B is canonically isomorphic to

$$\{y_0\} \times S^1 \cup [M^n \times \{11/12\}]$$

and B is a strong deformation retract of A. Note that since the PD isomorphism

$$F : M^n \times I \to M^n \times I$$

satisfies $F|A'$ =identity, then the PD isomorphism

$$H : (M^n \times S^1)_0 \to X_2^{n+1}$$

induced by F satisfies $H|A$ =identity.

Next, surgery will be used. References for this are [Milnor 6] and [Kervaire-Milnor].

In order to apply theorem 2b) surgery will be performed to $(M^n \times S^1)_0$ and X_2^{n+1} to kill the fundamental groups and get simply connected manifolds W_1^{n+1} and W_2^{n+1} such that $\partial W_1^{n+1} = S^n$ and $\partial W_2^{n+1} = \Sigma^n$.

It will be proved that the surgery can be performed inside A so that the PD isomorphism H 'extends' to a PD isomorphism $W_1^{n+1} \to W_2^{n+1}$. Also, it will be shown that if M^n is a spin manifold then W_1^{n+1} (and therefore W_2^{n+1}) is also a spin manifold, if the surgery is performed carefully enough. Choose as base point $* = [(y_0, 11/12)]$.

Lemma 19: *The inclusion* $i_B : B \to (M^n \times S^1)_0$ *induces an epimorphism*

$$\pi_1(B) \to \pi_1((M^n \times S^1)_0)$$

Proof: $(M^n \times S^1)_0 = M^n \times S^1 -$open disk. Since $n \geq 7$ (in fact $n \geq 2$ suffices), the inclusion

$$(M^n \times S^1)_0 \subseteq M^n \times S^1$$

induces an isomorphism

$$\pi_1((M^n \times S^1)_0) \to \pi_1(M^n \times S^1)$$

Henceforth, to prove the Lemma it suffices to prove that

$$\pi_1(B) \to \pi_1(M^n \times S^1)$$

is an epimorphism. But $\pi_1(B)$ is the free product of $\pi_1(M^n) \times \{11/12\}$ and $\pi_1(\{y_0\} \times S^1)$, and the diagram

$$M^n \times \{11/12\} \overset{\longleftarrow}{\longrightarrow} M^n \times S^1 \overset{\longleftarrow}{\longrightarrow} \{y_0\} \times S^1$$

(the arrows are inclusions and projections) gives a direct product diagram

$$\pi_1(M^n \times \{11/12\}) \overset{\longleftarrow}{\longrightarrow} \pi_1(M^n \times S^1) \overset{\longleftarrow}{\longrightarrow} \pi_1(\{y_0\} \times S^1)$$

therefore $\pi_1(B) \to \pi_1(M^n \times S^1)$ is an epimorphism. Q.E.D.

LEMMA 20: *Any map $S^1 \to M^n \times S^1$ is homotopic to an embedding $S^1 \to A$.*

PROOF: By Lemma 19, any map $S^1 \to M^n \times S^1$ is homotopic to a map $S^1 \to B \subseteq A$. Since $n \geq 7$ and A is open, any map $S^1 \to B$ is homotopic to an embedding $S^1 \to A$. Q.E.D.

Let S^1_i, $i = 1, \ldots, r$ be copies of the circle S^1, and $\beta_i : S^1_i \to A \subseteq (M^n \times S^1)_0$ embeddings representing a set of generators of $\pi_1((M^n \times S^1)_0)$. Since $n \geq 7$, we can assume the embeddings have disjoint images. Let $\tilde{\beta}_i : S^1_i \to A \subseteq (M^n \times S^1)_0$ be a "thickening."of β_i, i.e., $\tilde{\beta}_i(x, 0) = \beta_i(x)$. These thickenings exist since the normal bundle of $\beta_i(S^1_i)$ in A is trivial (manifolds are orientable). Also, assume the $\tilde{\beta}_i$ have disjoints images. Then $H \circ \tilde{\beta}_i : S^1_i \to A \subseteq X^{n+1}_2$ are embeddings, which represent a set of generators of $\pi_1(X^{n+1}_2)$, and with thickenings $H \circ \tilde{\beta}_i$. The trace of the surgery on $(M^n \times S^1)_0$ based on the embeddings $\tilde{\beta}_i$ is the manifold Z^{n+2}_1 obtained from

$$D^2_1 \times D^n \cup \cdots \cup D^2_r \times D^n + (M^n \times S^1)_0 \times I$$

identifying $x \in S^1_i \times D^n \subseteq D^2_i \times D^n$ to $(\tilde{\beta}_i(x), 1) \in (M^n \times S^1)_0 \times \{1\} \subseteq (M^n \times S^1)_0 \times I$. Then Z^{n+2}_1 is a smooth manifold except along certain corners, and

$$\partial Z^{n+2}_1 = M^n \times S^1 \times \{0\} \cup S^1 \times I \cup W^{n+1}_1$$

where W^{n+1}_1 is a 1-connected smooth manifold with $\partial W^{n+1}_1 = S^n \times \{1\}$. Similarly, there is a trace Z^{n+2}_2 of the surgery on X^{n+1}_2 along $H \circ \beta_i$ with

$$\partial Z^{n+2}_2 \cong X^{n+1}_2 \times \{0\} \cup \Sigma^n \times \cup W^{n+1}_1$$

where W^{n+1}_2 is a 1-connected smooth manifold with $\partial W^{n+1}_2 \cong \Sigma^n \times \{1\}$. The PD isomorphism $H : (M^n \times S^1)_0 \to X^{n+1}_2$ is the identity on A. Therefore

$$H \times 1_I : (M^n \times S^1)_0 \times I \to X^{n+1}_2 \times I$$

extends by the identity to give a PD isomorphism $Z^{n+2}_2 \to Z^{n+1}_2$. By restriction this gives a PD isomorphism $(W^{n+1}_1, \partial W^{n+1}_1) \to (W^{n+1}_2, \partial W^{n+1}_2)$. Therefore there is also a PL isomorphism between these manifolds.

Recall that an orientable manifold is spin if and only if it is 2-parallelizable. If M^n is spin, so is $M^n \times S^1$. Therefore $(M^n \times S^1)_0$ is also spin, therefore 2-parallelizable. But in [Milnor 6] it is shown that the thickenings $\tilde{\beta}_i$ can be chosen so that Z_1^{n+2} is 2-parallelizable and therefore spin. Therefore W_1^{n+1} is also spin. Since W_2^{n+1} is PL isomorphic to W_1^{n+1}, W_2^{n+1} is also spin.

THEOREM 21: *Let M^{8n+1} be a closed connected smooth manifold. If M^{8n+1} is spin then*

$$f_R(I_c(M^{8n+1})) = 0$$

PROOF: Let $\Sigma^{8n+1} \in I_c(M^{8n+1})$. Then there are 1-connected smooth manifolds W_1^{8n+2}, W_2^{8n+2} with $\partial W_1^{8n+2} \cong S^{8n+1}$, $\partial W_2^{8n+2} \cong \Sigma^{8n+1}$, and a PL isomorphism of pairs

$$(W_1^{8n+2}, \partial W_1^{8n+2}) \cong (W_2^{8n+2}, \partial W_2^{8n+2})$$

and applying PROPOSITION 2b) we get $f_R(\Sigma^{8n+1})$=Q.E.D.

4. The pairing $\bar{\rho}_{n,k}$

4.1.- Definition of the pairing and its properties

In this section we construct and study a bilinear map

$$\bar{\rho}_{n,k-1} : \Theta_n \times \pi_{n+k-1}(S^{n-1}) \to \Theta_{n+k}$$

which turns out to be an extension of the pairing

$$\tau_{n,k} : \Theta_n \times \pi_k(SO(n-1)) \to \Theta_{n+k}$$

of Milnor-Novikov (see [Lashof]).

The pairing $\bar{\rho}_{n,k-1}$ is in fact a stable version of the map $\rho_{n,k}$ of [Bredon].

We will relate $\bar{\rho}_{n,k-1}$ to concordance inertia groups and use the results of previous sections to prove some consequences.

Consider a manifold W^{n+k-1} and a framed submanifold $\langle M^k, F\rangle$. In case $\partial W^{n+k-1} \neq \emptyset$ and $\partial M^k \neq \emptyset$ we will require that $M^k \cap \partial W^{n+k-1} = \partial M^k$ the intersection being transverse. Let $g : D^{n-1} \to D^{n-1}$ be a diffeomorphism, such that g =identity on a neighborhood of $S^{n-2} = \partial D^{n-1}$. Take a closed tubular neighborhood T of M^k in W^{n+k-1}, and let

$$f : M^k \times D^{n-1} \to T$$

be a representation corresponding to the framing F. Let $g \circ \langle M^k, F\rangle$ be the following diffeomorphism of W^{n+k-1}:

On T let $g \circ \langle M^k, F\rangle$ be $f \circ g \circ f^{-1}$ and on $W^{n+k-1} - T$ let $g \circ \langle M^k, F\rangle$ be the identity. This makes sense since g =identity on a neighborhood of $S^{n-2} = \partial D^{n-1}$. It follows readily from the tubular neighborhood theorem that the concordance class of $g \circ \langle M^k, F\rangle$ is independent of the particular tubular neighborhood T, and representation f corresponding to F, chosen for the construction.

Suppose now that g' and g are diffeomorphisms of D^{n-1} which are concordant via a concordance

$$H : D^{n-1} \times I \to D^{n-1} \times I$$

with H =identity on a neighborhood of $S^{n-2} \times I$. Then one can carry H via $1_{M^k} \times f$ to a concordance between $g \circ \langle M^k, F\rangle | T$ and $g' \circ \langle M^k, F\rangle | T$ and this concordance can be extended by the identity to a concordance between $g \circ \langle M^k, F\rangle$ and $g' \circ \langle M^k, F\rangle$. So that the concordance class of $g \circ \langle M^k, F\rangle$ does not change if we change g within its concordance class.

If Σ^n is a homotopy sphere, let

$$g : S^{n-1} \to S^{n-1}$$

be a corresponding diffeomorphism. Let $\Sigma^n \circ \langle M^k, F\rangle$ be the concordance class of $g \circ \langle M^k, F\rangle$. This is well defined according to the above discussion.

If $\Sigma^n, \Sigma'^n \in \Theta_n$ are homotopy n-spheres corresponding to $g : D^{n-1} \to D^{n-1}$ and $g' : D^{n-1} \to D^{n-1}$ then $\Sigma^n \# \Sigma'^n$ corresponds with $g \circ g'$. One readily checks that

$$(g \circ g') \circ \langle M^k, F\rangle = (g \circ \langle M^k, F\rangle) \circ (g' \circ \langle M^k, F\rangle)$$

and this implies that

$$(\Sigma^n \# \Sigma'^n) \circ \langle M^k, F\rangle = (\Sigma^n \circ \langle M^k, F\rangle) \circ (\Sigma'^n \circ \langle M^k, F\rangle)$$

Let $\langle M^k, F\rangle$ and $\langle M'^k, F'\rangle$ be two framed submanifolds of W^{n+k-1} which are disjoint and well apart. Let $\langle M^k + M'^k, F + F'\rangle$ denote the framed submanifold which is the union of M and M'. Then it follows from the definition that

$$\Sigma^n \circ \langle M^k + M'^k, F + F'\rangle = (\Sigma^n \circ \langle M^k, F\rangle) \circ (\Sigma^n \circ \langle M'^k, F'\rangle)$$

Now we prove that if $\langle M^k, F\rangle$ and $\langle M'^k, F'\rangle$ are framed cobordant in W^{n+k-1} then

$$\Sigma^n \circ \langle M^k, F\rangle = \Sigma^n \circ \langle M'^k, F'\rangle$$

If $\langle V^{k+1}, G\rangle$ is a framed cobordism between M^k and M'^k then $\Sigma^n \circ \langle V^{k+1}, G\rangle$ gives a concordance between $\Sigma^n \circ \langle M^k, F\rangle$ and $\Sigma^n \circ \langle M'^k, F'\rangle$ as desired.

Let now W^{n+k-1} be S^{n+k-1}. If $M^k = S^k \subseteq S^{n+k-1}$, then we can identify $\langle S^k, F\rangle$ with an element $\hat{\alpha} \in \pi_k(SO(n-1))$, and

$$\Sigma^n \circ \langle S^k, F\rangle = \tau_{n,k}(\Sigma^n, \hat{\alpha})$$

where $\tau_{n,k} : \Theta_n \times \pi_k(SO(n-1)) \to \Theta_{n+k}$ is the Milnor-Munkres-Novikov pairing (see [Lashof]).

Finally observe that there is a natural one-to-one correspondence between elements of $\pi_{n+k-1}(S^{n-1})$ and framed cobordism classes of framed k-submanifolds of S^{n+k-1} (see [Stong]). If $\alpha \in \pi_{n+k-1}(S^{n-1})$ let $\langle M^k, F\rangle$ be the corresponding framed submanifold. Let

$$\bar{\rho}_{n,k-1} : \Theta_n \times \pi_{n+k-1}(S^{n-1}) \to \Theta_{n+k}$$

be defined by

$$\bar{\rho}_{n,k-1}(\Sigma^n, \alpha) = \Sigma^n \circ \langle M^k, F \rangle$$

Therefore we have that

$$\tau_{n,k}(\Sigma^n, \hat{\alpha}) = \bar{\rho}_{n,k-1}(\Sigma^n, J(\hat{\alpha}))$$

hence

THEOREM 22 : *There is a bilinear map*

$$\bar{\rho}_{n,k-1}(\Sigma^n, \alpha) = \Sigma^n \circ \langle M^k, F \rangle$$

such that the following diagram commutes

$$\begin{tikzcd}
\Theta_n \times \pi_{n+k}(SO(n-1)) \arrow[d, "1 \times J_k"'] \arrow[dr, "\tau_{n,k}"] & \\
\Theta_n \times \pi_{n+k-1}(S^{n-1}) \arrow[r, "\bar{\rho}_{n,k-1}"] & \Theta_{n+k}
\end{tikzcd}$$

In the next proposition it is proved that $\bar{\rho}_{n,k-1}$ is related to $\rho_{n,k}$ by suspension

PROPOSITION 23 : *Let $S : \pi_{n+k-1}(S^{n-1}) \to \pi_{n+k}(S^n)$ be the suspension homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd}
\Theta_n \times \pi_{n+k-1}(S^{n-1}) \arrow[r, "\bar{\rho}_{n,k-1}"] \arrow[d, "1 \times S"'] & \Theta_{n+k} \\
\Theta_n \times \pi_{n+k}(S^n) \arrow[ur, "\rho_{n,k}"'] &
\end{tikzcd}$$

PROOF: Represent $\alpha \in \pi_{n+k-1}(S^{n-1})$ by a framed submanifold $\langle M^k, F \rangle$ of S^{n+k-1}. Let ϵ be a normal vector field to S^{n+k-1} in S^{n+k}. Then $S(\alpha)$ corresponds to the framed manifold $\langle M^k, F + \epsilon \rangle$ in S^{n+k-1}. We can push $\langle M^k, F + \epsilon \rangle$ along $-\epsilon$, to obtain a framed cobordant (to $\langle M^k, F + \epsilon \rangle$) manifold $\langle M'^k, G \rangle \subseteq D_-^{n+k} \subseteq S^{n+k}$. Let T be a tubular neighborhood of M^k in S^{n+k-1} and let

$$\bar{F} : M^k \times D^{n-1} \to T$$

be a product representation corresponding to F.

Then using tubular neighborhood theory we can assume that $\langle M'^k, G\rangle$ has a tubular neighborhood N in S^{n+k}, with a product representation

$$\bar{G} : M'^k \times D^n \to N$$

which has the following properties:
a) $N \subseteq D_-^{n+k}$ with $N \cap S^{n+k-1} = T$
b) $\bar{G}|M'^k \times D_+^{n-1} = 1_{M^k} \times \lambda \circ \bar{F} \circ (1_{M^k} \times \lambda^{-1})$
where $\lambda : D^{n-1} \to D_+^{n-1} \subseteq S^{n-1}$ is defined on 1.1. Let $\Sigma^n \in \Theta_n$ and let $f : D^{n-1} \to D^{n-1}$ be a corresponding diffeomorphism. Define $\bar{f} : S^{n-1} \to S^{n-1}$ by

$$\bar{f}(x) = \begin{cases} \lambda \circ f \circ \lambda^{-1}(x) & \text{if} \quad x \in \lambda(D^{n-1}) \\ x & \text{if} \quad x \notin \lambda(D^{n-1}) \end{cases}$$

Then $\bar{\rho}_{n,k-1}(\Sigma^n, \alpha)$ is represented by the diffeomorphism

$$f \circ \langle M^k, F\rangle : S^{n+k-1} \to S^{n+k-1}$$

The homotopy sphere $\rho_{n,k}(\Sigma^n, S(\alpha))$ is obtained from the disjoint union

$$(S^{n+k} - \operatorname{int}(N)) + N$$

identifying

$$x \in \partial(S^{n+k} - \operatorname{int}(N)) = \partial N$$

to

$$\bar{G} \circ (1_{M^k} \times \bar{f}) \circ \bar{G}^{-1}(x) \in \partial N$$

In view of the homomorphisms of 1.1, to prove the Proposition it suffices to show that $\rho_{n,k}(\Sigma^n, S(\alpha))$ is the same than the homotopy sphere obtained from $D_-^{n+k} + D_+^{n+k}$ identifying

$$x \in S^{n+k-1} \subseteq \partial D_-^{n+k}$$

to

$$f \circ \langle M^k, F\rangle(x) \in S^{n+k-1} \subseteq \partial D_+^{n+k}$$

Observe that

$$D_-^{n+k} = \overline{(D_-^{n+k} - (\operatorname{int}(N) \cup T))} \cup N$$

Therefore the last identification above can be made in two steps:
First from $S^{n+k} - \mathrm{int}\,(N)$ to $\overline{(D_+^{n+k} - (\mathrm{int}\,(N) \cup T))} + D_+^{n+k}$ identifying

$$x \in \partial(D_+^{n+k} - \mathrm{int}\,(T)) \subseteq \overline{(D_+^{n+k} - (\mathrm{int}\,(N) \cup T))}$$

to

$$x \in D_+^{n+k}$$

and then identify

$$x \in S^{n+k} - \mathrm{int}\,(N) = \partial N$$

to

$$\begin{cases} f \circ \langle M^k, F \rangle & \text{if} \quad x \in T \subseteq \partial N \\ x & \text{if} \quad x \notin T \end{cases}$$

But this gives the same than $\rho_{n,k}(\Sigma^n, S(\alpha))$ Q.E.D.

COROLLARY 24: *The following diagram commutes for* $k-1 < n$ (see Prop. 4):

$$\begin{tikzcd}
\Theta_n \times \pi_{n+k}(SO(n-1)) \arrow[d, "1 \times J_k"'] \arrow[dr, "\tau_{n,k}"] & \\
\Theta_n \times \pi_{n+k-1}(S^{n-1}) \arrow[r, "\bar{\rho}_{n,k-1}"] \arrow[d, "1 \times S"'] & \Theta_{n+k} \arrow[dd, "p'"] \\
\Theta_n \times \pi_{n+k}(S^n) \arrow[ur, "\rho_{n,k}"'] \arrow[d, "p' \times 1"'] & \\
\mathrm{Coker} J_n \times \pi_{n+k}(S^n) \arrow[r, "c'"] & \mathrm{Coker} J_{n+k}
\end{tikzcd}$$

REMARK: The condition $k - 1 < n$ is needed in order to ensure the commutativity of the lower square (see 1.4.)

PROOF: This follows from Proposition 4, Theorem 22 and Proposition 23.

Remark: It has been mentioned that $\pi_n(PL/0) \cong \Theta_n$. Using this interpretation for Θ_n the pairing $\rho_{n,k}$ corresponds to the composition map

$$\pi_n(PL/0) \times \pi_{n+k}(S^n) \to \pi_{n+k}(PL/O)$$

defined by

$$([a],[b]) \to [a \circ b]$$

and the pairing $\bar{\rho}_{n,k-1}$ corresponds to

$$([a],[b']) \to [a \circ S(b')]$$

where $S(b')$ is the suspension of b'.

4.2.- Inertia groups and $\bar{\rho}_{n,k-1}$

Let W^{n+k} be a smooth closed manifold. Removing from W^{n+k} an open cell we obtain a smooth manifold W_0^{n+k} with $\partial W_0^{n+k} \cong S^{n+k-1}$.

Consider now the restriction map

$$r : [W_0^{n+k}, S^{n-1}] \to [S^{n+k-1}, S^{n-1}]$$

i.e. $r(g) = g|S^{n+k-1}$.

Theorem 25: $\bar{\rho}_{n,k-1}(\Theta_n \times r([W_0^{n+k}, S^{n-1}])) \subseteq I_c(W^{n+k})$

Proof: Let $\alpha \in \pi_{n+k-1}(S^{n-1})$, and represent α by a framed submanifold $\langle M^k, F\rangle$ of S^{n+k-1}. Then $\alpha \in r([W_0^{n+k}, S^{n-1}])$ if and only if there is a framed submanifold $\langle V_0^{k+1}, G\rangle$ of W_0^{n+k} such that

$$\partial V_0^{k+1} = V_0^{k+1} \cap S^{n+k-1} \text{ and } G|M^k = F$$

If Σ^n is a homotopy n-sphere, clearly $\Sigma^n \circ \langle V_0^{k+1}, G\rangle$ gives a diffeomorphism of W_0^{n+k} to itself which restricted to S^{n+k-1} is

$$\Sigma^n \circ \langle M^k, F\rangle = \bar{\rho}_{n,k-1}(\Sigma^n, \alpha)$$

Since diffeomorphisms of the disk are PD isotopic to the identity, carrying the isotopies to the tubular neighborhood of V_0^{k+1} in W_0^{n+k} via the framing, one sees that $\Sigma^n \circ \langle M^k, F\rangle$ is PD isotopic to the identity. Therefore by Proposition 16, $\bar{\rho}_{n,k-1}(\Sigma^n, \alpha) \in I_c(W^{n+k})$ whenever $\alpha \in r([W_0^{n+k}, S^{n-1}])$ Q.E.D.

Next we prove that a manifold W^{n+k} always exists such that $I_c(W^{n+k}) \supseteq \bar{\rho}_{n,k-1}(\Theta_n \times r([W_0^{n+k}, S^{n-1}]))$.

THEOREM 26: *For any n and $k \geq 1$ there exists an orientable $(n+k)$-manifold W^{n+k} such that*

$$I_c(W^{n+k}) \supseteq \text{image}(\bar{\rho}_{n,k-1})$$

PROOF: If $\Sigma^{n+k} \in I_c(W^{n+k})$ and $\Sigma'^{n+k} \in I_c(W'^{n+k})$, then $\Sigma^{n+k} \# \Sigma'^{n+k}$ belongs to $I_c(W^{n+k} \# W'^{n+k})$. This can be proved as follows.

Let $f : W^{n+k} \to W^{n+k} \# \Sigma^{n+k}$ be a diffeomorphism which is PD concordant to $1_{W^{n+k}}$. From Proposition 18 we can assume that there is a concordance H from f to $1_{W^{n+k}}$ and an open set $U \subseteq W^{n+k}$ such that H is the identity on $U \times I \subseteq W^{n+k} \times I$.

Similarly, there is a PD concordance H' from

$$f' : W'^{n+k} \to W'^{n+k} \# \Sigma'^{n+k}$$

to the identity, such that H' =identity on $U' \times I$ where $U' \subseteq W'^{n+k}$ is open.

If balls B, B' are removed from U and U' and the connected sum $W^{n+k} \# W'^{n+k}$ is formed, identifying ∂B to $\partial B'$, then there is a diffeomorphism

$$f \# f' : W^{n+k} \# W'^{n+k} \to (W^{n+k} \# \Sigma^{n+k}) \# (W'^{n+k} \# \Sigma'^{n+k})$$

defined by the conditions

$$f \# f' | (W^{n+k} - B) = f \qquad f \# f' | (W'^{n+k} - B') = f'$$

and there is a PD concordance H'' from $f \# f'$ to the identity defined by the conditions

Given $\alpha \in \pi_{n+k-1}(S^{n-1})$ we will construct a smooth closed connected and orientable manifold W_α^{n+k} such that if $(W_\alpha^{n+k})_0 = W_\alpha^{n+k}$- open disk, then $\alpha \in r([W_0^{n+k}, S^{n-1}]))$. Granting the existence of W^{n+k}, the proof goes as follows: Consider the set $\Theta_n \times \{\alpha\} \subseteq \Theta_n \times \pi_{n+k-1}(S^{n-1})$ and let P_α its image under $\bar{\rho}_{n,k-1}$, that is,

$$P_\alpha = \bar{\rho}_{n,k-1}(\Theta_n \times \{\alpha\}) \subseteq \Theta_{n+k}$$

By Theorem 25, $P_\alpha \subseteq I_c(W^{n+k})$. Let then

$$\alpha_1, \ldots, \alpha_r \in \pi_{n+k-1}(S^{n-1})$$

be such that

$$P_{\alpha_1} \cup \cdots \cup P_{\alpha_r} = \bar{\rho}_{n,k-1}(\Theta_n \times \pi_{n+k-1}(S^{n-1}))$$

Then

$$I_c(W_1^{n+k} \# \cdots \# W_r^{n+k}) \supseteq P_{\alpha_1} \cup \cdots \cup P_{\alpha_r}$$

Therefore if $W^{n+k} = W_1^{n+k} \# \cdots \# W_r^{n+k}$ then

$$I_c(W^{n+k}) \subseteq \bar{\rho}_{n,k-1}(\Theta_n \times \pi_{n+k-1}(S^{n-1}))$$

It remains to construct W^{n+k}.

Represent α by a framed submanifold $\langle M^k, F \rangle$ of S^{n+k-1} and let

$$\bar{F} : M^k \times D^{n-1} \to T$$

be a product representation, associated to F, of a tubular neighborhood T of M^k in S^{n+k-1}.

Since $k \geq 1$ and M^k is framed, there is some compact orientable manifold V_0^{k+1} such that $\partial V_0^{k+1} = M^k$, (recall that $\Omega_k^{fr} \to \Omega_k^{SO}$ is zero if $k \geq 1$). Consider now $T \subseteq S^{n+k-1} = \partial D^{n+k}$ and $M^k \times D^{n-1} \subseteq V_0^{k+1} \times D^{n-1}$. Form the disjoint union

$$V_0^{k+1} \times D^{n-1} + D^{n+k}$$

and identify

$$M^k \times D^{n-1} \subseteq V_0^{k+1} \times D^{n-1}$$

to

$$T \subseteq S^{n+k-1} = \subseteq D^{n+k}$$

via $\bar{F}$. After straightening corners along $M^k \times S^{n-2}$ a smooth manifold with boundary, X^{n+k}, is obtained such that there are natural embeddings

$$V^{k+1} \times D^{n-1} \xrightarrow{q} X^{n+k} \qquad D^{n+k} \xrightarrow{p} X^{n+k}$$

Consider two disjoint copies, X_+^{n+k}, X_-^{n+k}, of X^{n+k} and form the double

$$W_\alpha^{n+k} = D(X^{n+k})$$

of X^{n+k} (Cf. [Munkres 1]). Remove from $D(X^{n+k})$ the open ball $\operatorname{int} D^{n+k} \subseteq X_+^{n+k} \subseteq D(X^{n+k})$ and let $(W_\alpha^{n+k})_0$ be the resulting manifold. Then

$$\partial (W_\alpha^{n+k})_0 = S^{n+k-1}$$

and the framed submanifold $\langle M^k, F\rangle$ of S^{n+k-1} is the framed boundary of $\langle V_0^{k+1}, G\rangle$, where $V_0^{k+1} \subseteq (W_\alpha^{n+k})_0$ and the framing G is induced from the standard framing to V_0^{k+1} in $V_0^{k+1} \times D^{n-1} \subseteq (W_\alpha^{n+k})_0$. Therefore, by Thom-Pontriagin construction

$$\alpha \in r([W_0^{n+k}, S^{n-1}]))$$

Q.E.D.

Remark: Suppose that the manifold V_0^{k+1} is a spin manifold. Then $V_0^{k+1} \times D^{n-1}$ is also a spin manifold and so is D^{n+k}. Then the spin structures of $V_0^{k+1} \times D^{n-1}$ and of $D^{n+k} \subseteq X^{n+k}$ are compatible along $V_0^{k+1} \times D^{n-1} \cap D^{n+k} = M^k \times D^{n-1}$ since they are spin structures associated to the framings. Therefore X^{n+k} is a spin manifold ([Minor 5], 1). Similarly, $W_\alpha^{n+k} = D(X^{n+k})$ is spin.

Corollary 27: *If* $n + k \equiv -1(\mod 4)$ *then*

$$g_R(\bar{\rho}_{n,k-1}(\Theta_n \times \pi_{n+k-1}(S^{n-1})) = 0$$

Proof: This follows from Theorems 14 and 26. Q.E.D.

Consider now the subgroup

$$\pi' = \pi'_{n+k-1,n-1} \subseteq = \pi$$

consisting of classes which can be represented by an element $\langle M^k, F\rangle$ with M^k bounding some spin manifold. If $M^k = S^k$, $k > 2$, then $M^k = \partial D^{k+1}$ so that $J(\pi_k(SO(n-1)) \subseteq \pi'$.

Also, by Theorem 3 a) the index of π' in π is at most equal to 2. In fact, $\pi' = \pi$ whenever $k \equiv 3, 4, 5, 5, 7$ or $8 \mod 8$, and

$$[\pi' : \pi] = 2$$

if $k \equiv 1, 2 \mod 8$.

Theorem 28: *If* $n + k \equiv -1(\mod 4)$ *or* $n + k \equiv 1(\mod 8)$ *then*

$$f_R(\bar{\rho}_{n,k-1}(\Theta_n \times \pi')) = 0$$

Proof: Let $\Sigma^{n+k} \in \bar{\rho}_{n,k-1}(\Theta_n \times \pi)$; by Theorem 26 and the remark following it, there is a spin manifold W^{n+k} such that $\Sigma^{n+k} \in I_c(W^{n+k})$. By Theorem

17 in case $n+k \equiv -1 (\mod 4)$ and by Theorem 21 in case $n+k \equiv 1 (\mod 8)$, $f_R(\Sigma^{n+k}) = 0$.

COROLLARY 29: *If* $n+k \equiv -1 \mod 4$ *or* $n+k \equiv 1 \mod 8$ *the map*

$$p' : \Theta_{n+k} \to \mathrm{Coker} J_{n+k}$$

is one to one when restricted to $\bar{\rho}_{n,k-1}(\Theta_n \times \pi')$.

PROOF: This follows from the fact that the sequences

$$\begin{tikzcd} 0 \arrow[r] & bP_{r+1} \arrow[r, shift right] & \Theta_r \arrow[l, shift right, "f_R"'] \arrow[r, "p'"] & \mathrm{Coker} J_r \arrow[r] & 0 \end{tikzcd}$$

are split exact for $r = n+k$, when $n+k$ is as in the hypothesis, as follows from the results of [Brumfiel 1,2,3] mentioned in section 1.2 above.

5.- Proof of Proposition 18

Let B^n_r, $r > 0$, denote the n-cube of 'width' $2r$.

$$B^n_r = [-r, r] \times \overset{n}{\overbrace{\cdots}} \times [-r, r] \subseteq \mathbb{R}^n$$

B^n_1 is a cell complex with 2^n vertexes:

$$v_1, \ldots, v_{2^n}$$

where v_i has coordinates ± 1. It follows that B^n_r is a cell complex with 2^n vertexes:

$$rv_1, \ldots, rv_{2^n}$$

For each $i = 1, \ldots, n$ there is a pair of $(n-1)$-dimensional faces of B^n_r

$$B^n_{(i,-r)} = \{x \in B^n_r \mid x_i = -r\}, \quad B^n_{(i,+r)} = \{x \in B^n_r \mid x_i = r\}$$

Taking these together they include all the vertexes v_j and form a cell complex that is the boundary, ∂B^n_r, of the cube B^n_r.
If $0 < r' < r$ consider the annulus

$$A^n_{r',r} = B^n_r - \operatorname{int}(B^n_{r'})$$

Then $A^n_{r',r}$ is a cell complex with 2^{n+1} vertexes

$$rv_1, \ldots, rv_{2^n}, r'v_1, \ldots, r'v_{2^n}$$

The boundary of $A^n_{r',r}$ has two connected components: $\partial B^n_{r'}$, called the inner boundary, and ∂B^n_r, called the outer boundary; the vertexes $r'v_1, \ldots, r'v_{2^n}$ lie in the inner boundary and the vertexes $rv_1, \ldots, rv_{2^n}$ lie in the outer boundary.
Let $0 < s' < s$, then there is a vertex map from $A^n_{r',r}$ to $A^n_{s',s}$ which sends $r'v_i$ to $s'v_i$ and rv_i to sv_i. Triangulate $A^n_{r',r}$ without introducing new vertexes ([Hudson]). The above vertex map transports this triangulation of $A^n_{r',r}$ into a triangulation of $A^n_{s',s}$, and for these triangulations the vertex map extends linearly over each simplex to give a PL isomorphism

$$k_0 = k_0(r', r; s', s) : A^n_{r',r} \to A^n_{s',s}$$

Note that if x is in the inner boundary of $A_{r',r}$ then $k_0(x) = (s'/r')x$, and if x is in the outer boundary of $A_{r',r}$ then, $k_0(x) = (s/r)x$.

Lemma 30: *There is a PL isomorphism*

$$k : B_1^n \to B_2^n$$

with restrictions satisfying: $k|B^n_{1/2}$ *=identity; and* $k|\partial B_1^n$ *is equal to* $k(x) = 2x$.

Proof: Clearly $B_1^n = B^n_{1/2} \cup A^n_{1/2,3/4} \cup A^n_{3/4,1}$ and $B_2^n = B^n_{1/2} \cup A^n_{1/2,1} \cup A^n_{1,2}$. Define k as follows:

$$\begin{array}{rcl} k|B^n_{1/2} & = & \text{identity} \\ k|A^n_{1/2,3/4} & = & k_0(1/2,3/4;1/2,1) \\ k|A^n_{3/4,1} & = & k_0(3/4,1;1,2) \end{array}$$

This is the required map. Q.E.D.

Corollary 31: *There is a PL isomorphism*

$$k_1 : B_1^n \times I \to B_2^n \times I$$

such that $k_1|B^n_{1/2} \times I$*=identity and* $k_1|\partial B_1^n \times I$ *is given by* $k_1(x,t) = (2x,t)$

Proof: Take $k_1 = k \times 1_I$

Lemma 32: *There is a PL isomorphism*

$$k_2 : B_2^n \times I \to \partial B_1^n \times I \times I \cup B_1^n \times I \times \{1\}$$

such that $k_2|B_1^n \times I$ *is* $k_2(x,t) = (x,t,1)$ *and* $k_2|\partial B_2^n \times I$ *is given by* $k_2(x,t) = ((1/2)x, t, 0)$.

Proof: Let $k_2|B_1^n \times I$; we shall extend this to $A^n_{1,2} \times I$. The cell complex $A^n_{1,2} \times I$ has vertexes

$$v_1 \times \{0\}, \ldots, v_{2^n} \times \{0\}, 2v_1 \times \{0\}, \ldots, 2v_{2^n} \times \{0\}$$

$$v_1 \times \{1\}, \ldots, v_{2^n} \times \{1\}, 2v_1 \times \{1\}, \ldots, 2v_{2^n} \times \{1\}$$

On the other hand $B_1^n \times I \times I$ is a cell complex with vertexes

$$v_i \times \{j\} \times \{k\}, \quad j,k = 0 \text{ or } 1$$

The vertex correspondence

$$v_i \times \{j\} \to v_i \times \{j\} \times \{1\}, \quad 2v_i \times \{j\} \to v_i \times \{j\} \times \{0\}$$

extends as in page 40 to give a PL isomorphism

$$k' : A^n_{1,2} \times I \to \partial B^n_1 \times I \times I$$

such that one has

$$\begin{array}{ll} k'(x,t) = (x,t,1) = & \text{for } x \in \partial B^n_1 \\ k'(x,t) = (x,t,0) = & \text{for } x \in \partial B^n_2 \end{array}$$

Let $k_2|A^n_{1,2} \times I = k'$. Then k_2 is the required map. Q.E.D.
Let now

$$\begin{array}{rcl} T & = & \{(s,t) \in I \times I \mid t \leq 1-s\} \\ T' & = & \{(s,t) \in I \times I \mid t \geq 1-s\} \\ E & = & T \cap T' \end{array}$$

PROPOSITION 33: *Let $H : B^n_1 \times I \to B^n_1 \times I$ be a PL isomorphism, such that*

$$H|B^n_1 \times \{1\} = identity \qquad H(B^n_1 \times \{0\}) = B^n_2 \times \{0\}$$

then there is a PL isomorphism

$$\bar{H} : B^n_1 \times I \times I \to B^n_1 \times I \times I$$

such that $\bar{H}(x,t,0) = (H(x,t),0)$ and $\bar{H}|B^n_1 \times T' =$identity.
PROOF: Consider a triangulation K of $B^n_1 \times I$ with vertexes

$$(p_1,t_1),\ldots,(p_r,t_r) \quad p_i \in B_i, \quad t_i \in I$$

such that H is simplicial in this triangulation. Make $B^n_1 \times T$ into a cell complex as follows: Take as vertexes

$$(p_i,t_i,0) \qquad (p_i,0,t_i) \quad i = 1,\ldots,r$$

Take as k-cells of $B^n_1 \times T$ the cells spanned by

$$(p_{i_0},t_{i_0},0),\ldots,(p_{i_k},t_{i_k},0),(p_{i_0},0,t_{i_0}),\ldots,(p_{i_k},0,t_{i_k})$$

whenever $(p_{i_0},t_{i_0}),\ldots,(p_{i_k},t_{i_k})$ span a $k-1$-simplex of K.
Define now a map on the vertexes:
If H sends (v_i,t_i) to (v_j,t_j) let the vertex map send $(v_i,t_i,0)$ to $(v_j,t_j,0)$ and $(v_i,0,t_i)$ to $(v_j,0,t_j)$ This vertex map extends now to a PL isomorphism

$$\bar{H}' : B^n_1 \times T \to B^n_1 \times T$$

Note that $\bar{H}'|B_1^n \times E$ is the identity. Define now $\bar{H}$ by the conditions

$$\begin{array}{rcl} \bar{H}|B_1^n \times T & = & \bar{H}' \\ \bar{H}|B_1^n \times T & = & \text{identity} \end{array}$$

Q.E.D.

PROPOSITION 34: *Let $H : B_1^n \times I \to B_1^n \times I$ be as in previous Proposition. Then there is a PL isomorphism*

$$G : B_1^n \times I \to B_1^n \times I$$

such that

$$\begin{array}{rcl} G|\,\partial B_1^n \times I & = & H|\,\partial B_1^n \times I \\ G|\,B_1^n \times \{1\} \cup B_{1/2}^n \times I & = & \text{identity} \end{array}$$

PROOF: Let $\bar{H}$ be as in previous Proposition, and let

$$\begin{array}{l} k_1 : B_1^n \times I \to B_2^n \times I \\ k_2 : B_2^n \times I \to \partial B_1^n \times I \times I \cup B_1^n \times I \times \{1\} \end{array}$$

be as in Corollary 31 and in Lemma 32. Let also

$$G = k_1^{-1} \circ k_2^{-1} \circ H \circ k_2 \circ k_1$$

Since $\bar{H}$ =identity on

$$B_1^n \times I \times \{1\} \cup \partial B_1^n \times \{1\} \times I \subseteq B_1^n \times T'$$

it follows that G is the identity on

$$k_1 \circ k_2(B_1^n \times I \cup B_{1/2}^n \times I)$$

And since $\bar{H}$ is the identity on

$$k_2 k_1(B_1^n \times \{1\} \cup B_{1/2}^n \times I) = B_1^n \times I \times \{1\} \cup \partial B_1^n \times \{1\} \times I \subseteq B_1^n \times T'$$

it follows also that G is the identity on

$$B_1^n \times \{1\} \cup B_{1/2}^n \times I \quad XxXx$$

Let

$$H(x,t) = (H_1(x,t), H_2(x,t)) \in B_1^n \times I$$

then if $x \in \partial B_1^n$ we have

$$\begin{aligned} G|\,\partial B_1^n \times I &= H|\,\partial B_1^n \times I \\ G|\,B_1^n \times \{1\} \cup B_{1/2}^n \times I &= \text{identity} \end{aligned}$$

Q.E.D.

COROLLARY 35: *Let N be a PL n-ball, and*

$$h : N \times I \to N \times I$$

a PL isomorphism such that

$$h(N \times \{0\}) = N \times \{0\} \qquad h|\,N \times \{1\} = \text{identity}$$

If $f : B_1^n \times I \to N \times I$ is any PL isomorphism such that $f : B_1^n \times \{i\} = N \times \{i\}$, $i = 1, 2$ then there is a PL isomorphism

$$g : N \times I \to N \times I$$

such that

$$\begin{aligned} g|\,\partial N \times I &= h|\,\partial N \times I \\ g|\,f(B_{1/2} \times I) \cup N \times I &= \text{identity} \end{aligned}$$

PROOF: Let $H = f^{-1} \circ h \circ f$. One checks that the hypothesis of Proposition 34 are verified by H. Let G be obtained from H as in Proposition 34. Then $g = f \circ h \circ f^{-1} : N \times I \to N \times I$ is the required PL isomorphism. Q.E.D.

Let $\Sigma^n \in I_c(M^n)$ and recall the following notation: $\varphi : D^n \to M^n$ is a smooth embedding used to form $M^n \# \Sigma^n$ as in page 18, and $\psi : D^n \to M^n$ is also a smooth embedding, verifying $\varphi(D^n) \cap \psi(D^n)$ and $y_0 = \psi(0)$. $f : M^n \to M^n \# \Sigma^n$ is an i-diffeomorphism and $H_0 : M^n \times I \to M^n \# \Sigma^n \times I$ is a PD concordance from f to 1.

To prove Proposition 18 the following six steps will be caried over:

Step 1: Modify f and H so that the paths $t \to (y_0, t)$ and $t \to H(y_0, t)$ are homotopic relative endpoints.

Step 2: Modify H further to ensure that $\bar{f} = H|M^n \times \{1/3\}$ is a PL isomorphism and $H|M^n \times [0, 1/3]$ is a PD concordance from f to $\bar{f}$, such that H =identity near $\{y_0\} \times [0, 1/3]$.

Step 3: Make H a PL isomorphism on $M^n \times [1/3, 1]$.

Step 4: Apply PL isotopy theorems to make $H|\{y_0\} \times I =$ identity.

Step 5: Apply regular neighborhood properties so that $H(N \times I) = N \times I$, for some PL ball N, $y_0 \in N$.

Step 6: Apply Corollary 35 to make $H =$ identity near $\{y_0\} \times I$, and adjust the parameter $t \in [0, 1]$.

The remaining of this section provides the details required for these steps.

LEMMA 36: *Let $b : [0,1] \to M^n \# \Sigma^n$ be a closed smooth path, $b(0) = b(1) = y_0$, such that*

$$b([0,1]) \cap \varphi(D^n) = \emptyset$$

Then there is a smooth isotopy

$$H_1 : (M^n \# \Sigma^n) \times I \to (M^n \# \Sigma^n) \times I$$

such that

$$H_1(x, 1) = (x, 1) \ \ \textit{for all} \ \ x \in M^n \# \Sigma^n$$

and

$$H_1(x, t) = (x, t)$$

for all x in some neighborhood of $\varphi(D^n)$ and for all $t \in [0, 1]$

$$H_1(y_0, t) = (b(t), t)$$

PROOF: Consider $b : [0,1] \to (M^n \# \Sigma^n) - \varphi(D^n)$ as an isotopy of the 0-dimensional submanifold $\{y_0\} \subseteq M^n \# \Sigma^n$ and apply the isotopy extension theorem of Milnor, to obtain an isotopy $\bar{H}_1$ on $(M^n \# \Sigma^n) - \varphi(D^n)$ which fixes points outside a compact set. Extend $\bar{H}_1$ via the identity in order to obtain the isotopy H_1on $M^n \# \Sigma^n$. Q.E.D.

LEMMA 37: *Let $\Sigma^n \in I_c(M^n$ then there is an i-diffeomorphism $f : M^n \to M^n \# \Sigma^n$ and a PD concordance*

$$H_2 : M^n \times I \to (M^n \# \Sigma^n) \times I$$

such that

$$t \to (y_0, t) \quad \textit{and} \quad t \to H_2(y_0, t)$$

are paths which are homotopic relative endpoints.

PROOF: $H_0 : M^n \times I \to (M^n \# \Sigma^n) \times I$ be as in page 44. Consider the path $\bar{b}(t) = H_0(y_0, 1-t)$, which is not in general smooth but can be approximated by a smooth path $b(t)$ homotopic to $\bar{b}$.

Apply Lemma 36 to $b(t)$, to obtain an isotopy H_1 such that

$$H_1(x,1) = (x,1)$$

for all $x \in M^n \# \Sigma^n$

$$H_1(x,t) = (x,t)$$

for all x in some neighborhood of $\varphi(D^n)$ and for all t, and

$$H_1(y_0,t) = (b(t),t)$$

for all t. Define now

$$H_2(x,t) = \begin{cases} H_0(x, 2t-1) & \text{for} \quad 1/2 \le t \le 1 \\ H_1(f(x), 2t) & \text{for} \quad 0 \le t \le 1/2 \end{cases}$$

Then h_2 is a PD concordance and $t \to H_2(y_0,t)$ is the composition of the path

$$t \to H_0(x,t) = (\bar{b}^{-1}(t), t)$$

with

$$t \to (b(t), t)$$

Since b is homotopic to $\bar{b}$, it follows that

$$t \to H_2(y_0, t)$$

is homotopic to

$$t \to (y_0, t)$$

relative endpoints. Q.E.D.

LEMMA 38: *With the same hypothesis as in previous Lemma, there is a PD concordance H_3 satisfying besides the properties of H_2 the following: $H_3(X,0) = (X,0)$ for all $x$in some neighborhood of y_0.*

PROOF: By continuity of H_2 and the fact that $H_2(y_0,0) = (y_0,0)$ there is some $\epsilon > 0$ such that

$$H_2(\psi(\epsilon D^n), 0) \subseteq \psi(D^n) \times \{0\}$$

Applying Palais-Cerf Lemma, there is a smooth isotopy

$$G : (M^n \# \Sigma^n) \times I \to (M^n \# \Sigma^n) \times I$$

such that G is the identity outside $\psi(D^n) \times I$,

$$\begin{array}{lll} G(x,1) = (x,1) & \text{for} & x \in M^n \# \Sigma^n \\ G(H_2(x,0)) = (x,0) & \text{for} & x \in \psi(\epsilon D^n) \end{array}$$

Define H_3 as $H_3(x,t) = G \circ H_2(x,t)$ for all x and all t. Since G keeps everything fixed outside $\psi(D^n) \times I$ it follows that

$$t \to H_3(y_0,t) \qquad t \to H_2(y_0,t)$$

are homotopic relative endpoints and the other properties of H_3 are straightforward. Q.E.D

LEMMA 39: *Let $f : M^n \to M^n \# \Sigma^n$ be an i-diffeomorphism and let $\alpha : K \to M^n$ be a smooth triangulation. Then there is a PL isomorphism (on α)*

$$\bar{f} : M^n \to M^n \# \Sigma^n$$

and a PD isotopy H_4 from $\bar{f}$ to f such that H_4 is the identity on $\psi(D^n) \times I$.

PROOF: Consider $f \circ \alpha : K \to M^n \# \Sigma^n$ and $\alpha : K \to M^n \# \Sigma^n$. These maps are smooth triangulations. Since $f|\psi(D^n)$ =identity, it follows that $\alpha^{-1} \circ f \circ \alpha : K \to K$ is the identity on $\alpha^{-1}\psi(D^n)$, in particular $\alpha^{-1} \circ f \circ \alpha$ is PL on $\alpha^{-1}\psi(D^n)$. According to [Munkres 1] there is, for any $\delta > 0$ a δ-approximation to α

$$\beta :\to M^n \# \Sigma^n$$

which is a smooth triangulation satisfying

$$\beta|\alpha^{-1}\psi(D^n) = \alpha|\alpha^{-1}\psi(D^n)$$

and $\beta^{-1} \circ f \circ \alpha : K \to K$ is a PL isomorphism. Define

$$\bar{f} = \alpha \circ \beta^{-1} \circ f : M^n \to M^n \# \Sigma^n$$

Since

$$\alpha^{-1} \bar{f} \alpha = \beta^{-1} f \alpha$$

$\bar{f}$ is certainly a PL-isomorphism.

Consider now the smooth maps

$$f \circ \alpha : K \to M^n \# \Sigma^n$$

and

$$\bar{f} \circ \alpha : K \to M^n \# \Sigma^n$$

then

$$\bar{f} \circ \alpha | \alpha^{-1}(\psi(D^n)) = f \circ \alpha | \alpha^{-1}(\psi(D^n))$$

Let $\delta' > 0$. If δ in the paragraph above is small enough, then $\bar{f}$ is a δ'-approximation to f. Applying now [Munkres 1] 10.9, provable in a relative form using [Munkres 1] 5.15, there is a smooth isotopy

$$\bar{F} : K \times I \to (M^n \# \Sigma^n) \times I$$

satisfying

$$\begin{array}{rcl} \bar{F}(x,0) & = & (\bar{f}(x),0) \\ \bar{F}(x,1) & = & (f \circ \alpha(x,1) \end{array}$$
$$\bar{F}|\alpha^{-1}(\psi(D^n)) \times I = \alpha \times 1_I$$

Define H_4 by

$$H_4(y,t) = \bar{F}(\alpha^{-1}(y),0)$$

then

$$\begin{array}{rcl} H_4(y,0) & = & \bar{F}(\alpha^{-1}(y),0) \\ & = & \bar{f} \circ \alpha \circ \alpha^{-1}(y,0) \\ & = & (f(y),0) \end{array}$$

and

$$\begin{array}{rcl} H_4(y,1) & = & \bar{F}(\alpha^{-1}(y),1) \\ & = & \bar{f} \circ \alpha \circ \alpha^{-1}(y,1) \\ & = & (f(y),1) \end{array}$$

and for $y \in \psi(D^n)$

$$H_4(y,t) = \bar{F}(\alpha^{-1}(y),t)$$

Q.E.D.

LEMMA 40: *Let $\Sigma^n \in I_c(M^n)$. Then there is an i-diffeomorphism $f : M^n \to M^n \# \Sigma^n$ and a concordance H_5 from f to the identity satisfying*

$$\begin{array}{l} H_5|M^n \times [1/3,1] \quad \textit{is a PL isomorphism} \\ H_5|\psi() \times [0,1/3] = \textit{identity} \end{array}$$

and the paths

$$\begin{array}{ll} t \to (y_0, t) & 1/3 \leq t \leq 1 \\ t \to H_5(y_0, t) & 1/3 \leq t \leq 1 \end{array}$$

considered as paths in $(M^n \# \Sigma^n) \times [1/3, 1]$ *are homotopic relative endpoints.*

Proof: Let H_3 be the concordance from an i-diffeomorphism f to the identity constructed in Lemma 38 and let H_4 be the concordance constructed for f in Lemma 39. Define H_5 by

$$H_5(x,t) = \begin{cases} H_4(x, 3t) & \text{if } \ 0 \leq t \leq 1/3 \\ H_4(x, 2-3t) & \text{if } \ 1/3 \leq t \leq 2/3 \\ H_3(x, 3t-2) & \text{if } \ 2/3 \leq t \leq 1 \end{cases}$$

one checks directly that H_5 has the required properties. Q.E.D.

Lemma 41: *Let* $\Sigma^n \in I_c(M^n)$*. Then there is an i-diffeomorphism* $f : M^n \to M^n \# \Sigma^n$ *and a concordance* H_6 *from* f *to the identity such that* $H_6(y_0, t) = (y_0, t)$ *for all* t *and* H_6 *is* PL *in a neighborhood of* $\{y_0\} \times I$*.*

Proof: Let H_5 be as in Lemma 46, then consider $H_5|M^n \times 1/3, 1]$ which is a PD isomorphism. It follows from section 3.1 that there is a PL isomorphism

$$G_1 : M^n \times [1/3, 1] \to M^n \# \Sigma^n \times [1/3, 1]$$

which is a δ-approximation to H_5 and such that

$$G_1 | M^n \times [1/3, 1] = H_5$$

Let $\bar{H}_5 : M^n \times [0,1] \to M^n \times [0,1]$ be defined as

$$\begin{array}{rcl} \bar{H}_5 | M^n \times [0, 1/3] & = & H_5 \\ \bar{H}_5 | M^n \times [1/3, 1] & = & G \end{array}$$

If δ is small, the PL paths

$$t \to \bar{H}_5(y_0, t) \qquad t \to H_5(y_0, t)$$

are homotopic in $M^n \# \Sigma^n \times [1/3, 1]$ relative endpoints. Therefore,

$$t \to \bar{H}_5(y_0, t) \qquad t \to (y_0, t)$$

are PL-homotopic in $M^n \# \Sigma^n \times [1/3, 1]$ relative endpoints. Since $\dim M^n \geq 3$ these paths are PL isotopic relative endpoints ([Hudson]). Therefore by the PL-isotopy extension theorem there is a PL isotopy

$$H_7 : (M^n \# \Sigma^n) \times [1/3, 1] \to (M^n \# \Sigma^n) \times [1/3, 1]$$

such that H_7 =identity on

$$(M^n\#\Sigma^n)\times\{1/3,1\}\times[0,1]\cup(M^n\#\Sigma^n)\times[1/3,1]\times\{1\}\cup\psi(D^n)\times[1/3,1]\times[0,1]$$

and

$$H_7(y_0,t,0)=(\bar{H}_5(y_0,t),0),\qquad 1/3\le t\le 1$$

Define $(H_7)_1$ by $((H_7)_1(x,t),0)=H_7(x,t,0)$. Let

$$H_6(x,t)=\begin{cases} H_5(x,t) & \text{if}\quad 0\le t\le 1/3\\ (H_7)_1^{-1}\bar{H}_5(x,t) & \text{if}\quad 1/3\le t\le 1\end{cases}$$

Then H_6 is a PD concordance from f to 1 which is the identity $\{y_0\}\times I$. Since H_6 is PL on $M^n\#\Sigma^n\times[1/3,1]$ and H_6 =identity on $\psi(D^n)\times[1/3,1]$ the Lemma follows. Q.E.D.

LEMMA 42: *Let $\Sigma^n\in I_c(M^n)$, then there is a concordance $H_8 : M^n\times I\to (M^n\#\Sigma^n)\times I$ from an i-diffeomorphism f to the identity and a PL ball $N\subseteq\psi(D^n)$ with $y_0\in\operatorname{int}(N)$, such that $H_8(N\times I)=N\times I$ and $H_8|\{y_0\}\times[0,1]$ =identity.*

PROOF: Let H_6 be as in previous Lemma. By continuity of H_6 there is some PL ball N with $y_0\in\operatorname{int}(N)\subseteq\psi(D^n)$ such that

$$H_6(N\times[0,1])\subseteq\operatorname{int}(\psi(D^n)\times[0,1])$$

Since H_6 is PL on $\psi(D^n)\times[0,1]$ we have that $H_6(N\times[1/3,1])$ is a regular neighborhood of $\{y_0\}\times[1/3,1]$ which meets $(M^n\#\Sigma^n)\times[1/3,1]$ along $N\times[1/3,1]$ and $N\times\{1\}$. By uniqueness of regular neighborhoods, ([Hudson], page) we can modify $H_6|N\times[1/3,1]$ via a PL isotopy to obtain a new PL isotopy relative $\partial(M^n\times[1/3,1])$, in order to obtain a new PL isomorphism

$$\bar{H}_6 : M^n\times[1/3,1]\to(M^n\#\Sigma^n)\times[1/3,1]$$

such that

$$\begin{aligned}\bar{H}_6|\partial(M^n\times[1/3,1]) &= H_6\\ \bar{H}_6|N\times[1/3,1] &= N\times[1/3,1]\\ \bar{H}_6|\{y_0\}\times[1/3,1] &= \text{identity}\end{aligned}$$

Define $H_8 : M^n\times I\to(M^n\#\Sigma^n)\times I$ by

$$\begin{aligned}H_8|M^n\times[0,1/3] &= H_6\\ H_8|M^n\times[1/3,1] &= \bar{H}_6\end{aligned}$$

Then $H_8(N \times [0,1]) = N \times [0,1]$ and $H_8| = \{y_0\} \times [0,1]$ =identity. Making an appropriate change of parameter t, it can be assured that $H_8|M^n \times [0,1/3] = f \times 1_{[0,1]}$ and $H_8|M^n \times [2/3,1]$=identity.

Proof of Proposition 18: Let H_8 be as in Lemma 42. Then $h = H_8|N \times [0,1]$ satisfies the hypothesis of Corollary 35. Let g be the PL isomorphism constructed from h in Corollary 35, and let $U \subseteq N$ be a neighborhood of y_0 such that $g|U \times I$ =identity.

Define H as

$$\begin{array}{lcl} H|N \times [0,1] & = & g \\ H|(M^n - N) \times [0,1] & = & H_8 \end{array}$$

Then $H|U \times [0,1] \cup M^n \times [2/3,1]$ =identity and $H|M^n \times \{0\}$ is an i-diffeomorphism. Q.E.D.

6. Low dimensional cases

6.1.- Using the exact sequence

$$0 \to I_c(M^n) \to \Theta_n \to [M^n, PL/O]$$

and a Postnikov tower for $PL/0$ it can be proved that $I_c(M^n) = 0$ for $n = 7$ and for $n = 8$.

6.2.- In dimension $n = 9$, Θ_9 has 8 elements. Applying Theorem 21, it follows that for any spin manifold M^9, $I_c(M^9)$ does not contain the Kervaire sphere Σ_0^9.

In case M^9 is not spin we do not know the answer. Also, since $\rho_{8,1} \neq 0$ ([Bredon]), $\bar{\rho}_{8,0} \neq 0$ and Theorem 26 implies that there is a manifold W^9 with $I_c(W^9) \supseteq \bar{\rho}_{8,0} \neq 0$.

6.3.- In dimension $n = 11$, since $bP_{11} = \Theta_{11} \cong \mathbb{Z}_{910}$ Theorem 17 implies that for any spin manifold M^{11}, $I_c(M^{11}) = 0$. In case M^{11} is not spin, again Theorem 17 implies that $I_c(M^{11})$ has at most $t_{11}/t'_{11} = 32$ elements; we do not know if an M^{11} exists such that $I_c(M^{11}) \neq 0$. Also, since $\mathrm{Coker}J_{11} = 0$ Theorem 28 implies that $\bar{\rho}_{n,k-1} = 0$ whenever $n+k = 11$ and $\pi'_{n+k-1,n-1} = \pi_{n+k,n-1}$ (see section 4.2).

6.4.- In dimension $n = 15$, $\mathrm{Coker}J_{15} \cong \mathbb{Z}_2$, which implies by Theorem 17 that for any spin manifold M^{15}, $I_c(M^{15})$ has at most 2 elements, and the Milnor sphere is not in $I_c(M^{15})$.

In case M^{15} is not spin, since $g_R(\Sigma_0^{15}) = 1 \neq 0$, Theorem 17 implies that $I_c(M^{15})$ does not contain Σ_0^{15}, and in fact, $I_c(M^{15})$ has at most $t_{15}/t_{15} = 128$ elements. Again, we do not know if $I_c(M^{15})$ can actually be that large.

Since $\rho_{14,1} \neq 0$ ([Bredon]), $\bar{\rho}_{14,0} \neq 0$, and there is a manifold W^{15} with $I_c(W^{15}) \supseteq \Rightarrow\ \bar{\rho}_{14,0} \neq 0$.

6.5.- For $n = 17$, Θ_{17} has 16 elements and $\mathrm{Coker}J_{17}$ has 8 elements. Therefore, Theorem 21 implies that $I_c(M^{17})$ has at most 2 elements if M^{17} is spin. Since $\rho_{16,1} \neq 0$ (ibid), $\bar{\rho}_{16,0} \neq 0$ and Theorem 25 implies that there is a manifold W^{17} with $I_c(W^{17} \supseteq \mathrm{image}\bar{\rho}_{16,0} \neq 0$.

The data used about Θ_n, bP_{n+1} and $\mathrm{Coker}J_n$ are taken from [Kervaire-Milnor].