G^2 Composite Cubic Bézier Curves

ABSTRACT

We present an algorithm for creating planar G^2 spline curves using rational Bézier cubic segments. The splines interpolate a sequence of points, tangents and curvatures. In addition each segment has two more geometric shape handles. These are obtained from an analysis of the singular point of the curve. The individual segments are convex, but zero curvature can be assigned at a junction point, hence inflection points can be placed where desired but cannot occur otherwise.

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1. Introduction

The purpose of this paper is to describe a unifying method to create cubic Bézier curve segments that satisfy contact conditions at both endpoints. The contact conditions specify tangents and curvatures. There are a number of partial solutions to this problem already described in the CAGD literature. In (DeBoor, Höllig and Sabin, 1987) it is shown how to find as many as three polynomial Bézier cubics that interpolate endpoints with specified contact conditions. In (Farin, 1993) it is shown how to find the weights for a rational Bézier curve if the intermediate control points have somehow already been fixed. This method, however, does not allow the curvature at either end to be zero, so that inflection points must be interior points and therefore are difficult to place. A more serious drawback is that the curves are controlled by control points rather than parameters with direct geometric meaning for the curve.

The curves described in this paper can profitably be compared to rational quadratic Bézier curves and splines (Farin, 1993). A quadratic Bézier curve is a conic arc in a control triangle. A spline created from them lies in a control polygon composed of triangles with collinear edges as in Figure 1.

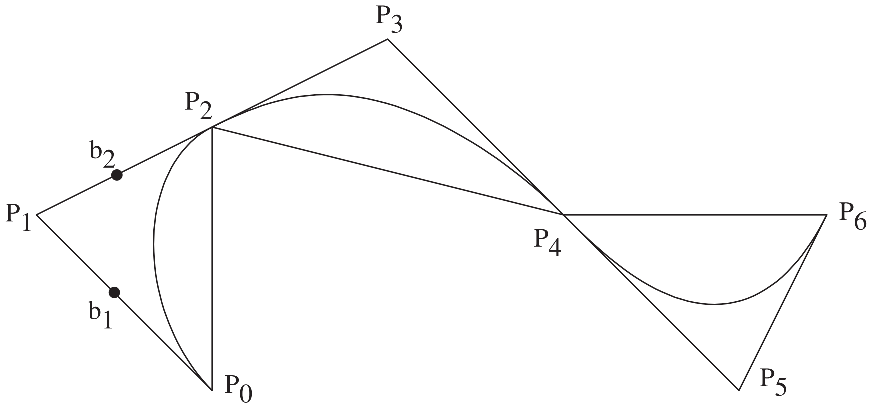

Figure 1. A G^2-cubic interpolating spline.

A particular conic arc within a triangle is usually determined by a weight w_1, which controls how far the curve is pulled toward the vertex, but could also be specified by giving a point inside the triangle that must be interpolated. Using conics however, there are only enough degrees of freedom to assign the curvature at one point of a spline. Also since a conic arc never has zero curvature, this prevents placing any inflection points. Consequently

G^2-quadratic splines are of limited usefulness.

The curves to be studied here are similar to quadratic curves and splines in that we use the same control polygon, working in a sequence of triangles. This differs from the usual cubic control polygon that allows free placement of control points. Now $\mathbf{b_1}$ is restricted to lie on the segment $\mathbf{P_0P_1}$ and $\mathbf{b_2}$ to lie on the segment $\mathbf{P_1P_2}$ (see Figure 1). The advantage of this is that it guarantees that no inflection point can occur in the interior of a triangle.

By using rational *cubic* Bézier curve segments in each triangle, we now have enough degrees of freedom to specify curvatures at both endpoints of every Bézier segment or every junction point of a spline. Furthermore, our construction allows the curvature to be set to zero at either or both ends of a curve segment. Thus an inflection point can be inserted into a spline at a junction point, such as $\mathbf{P_4}$ in Figure 1, but must be placed there purposely.

After the curvatures are set, the remaining degrees of freedom of the Bézier cubic within each triangle are controlled by two additional shape handles. The first is a point $\mathbf{B_0}$ within the triangle that is interpolated. If $\mathbf{B_0}$ is constrained to the line joining $\mathbf{P_1}$ and the midpoint of $\mathbf{P_0P_2}$, moving it has the same effect as varying the weight w_1 in the quadratic case. However as we shall see, $\mathbf{B_0}$ cannot lie too near the vertex $\mathbf{P_1}$ if the curvatures at $\mathbf{P_0}$ and $\mathbf{P_2}$ are large. The second control is by means of a slider that varies the curves through $\mathbf{B_0}$ within a family. The relationship of the slider to the slope at $\mathbf{B_0}$ is observed to be monotonic most of the time, but the exact dependence is not pursued in this paper. From the values given for these two shape handles, the locations of the control points $\mathbf{b_1}$ and $\mathbf{b_2}$ and the weights w_1 and w_2 are computed. The Bézier control points $\mathbf{b_0}$ and $\mathbf{b_3}$ coincide with $\mathbf{P_0}$ and $\mathbf{P_2}$, respectively. The weights w_0 and w_3 have value 1.

These two controls are fully local in the sense that curvatures at the endpoints are not affected by them. The user can raise or lower the curve by moving $\mathbf{B_0}$ within the triangle and then slide through the entire family of rational cubic Bézier curves that pass through $\mathbf{B_0}$ and join the endpoints with prescribed tangents and curvatures.

The controls are obtained from a study of the singular point of the cubic. That means ultimately we need formulas for the control points and weights of a Bézier curve in terms of the curvatures at the ends and the singular point of the curve. These formulas are derived in Section 3. In Section 4 we see that the locus of singular points of Bézier curves with given interpolation conditions is a curve S of degree six. Certain arcs of S are composed of the singular points of cubics that are acceptable for modeling. These arcs are described in Section 5. Section 6 shows that the curve S is birationally equivalent to a curve Q of degree 4. Using Q, in Section 7 we supply a proof of most of the claims made in Section 5 about arcs of S. Section 8 is devoted to the case in which one or both of the curvatures at the endpoints is zero. Section 9 contains some details related to programming and a summary.

Section 2. Geometric Preliminaries

A triangle $\mathbf{P_0P_1P_2}$ defines two systems of coordinates. The first are barycentric for the affine plane. The barycentric coordinates of a point $\mathbf{P}$ are $\mathbf{P}(s,t,u)$, where $s+t+u=1$. The barycentric coordinates of the vertices of the triangle are $\mathbf{P_0}(1,0,0)$, $\mathbf{P_1}(0,1,0)$ and $\mathbf{P_2}(0,0,1)$. The second coordinates are homogeneous for the entire projective plane. The homogeneous coordinates of $\mathbf{P}$ are $\mathbf{P}[s,t,u]$, written with square brackets. The only condition now on s,t,u is that they cannot all be zero. If $\mathbf{P}$ does not lie at infinity its homogeneous coordinates can be normalized to barycentric coordinates by dividing by $s+t+u$.

The following properties of an irreducible cubic are all equivalent (Brieskorn and Knorrer, 1986):

- It is *rational*, in the sense that it has a rational parametrization.
- It can be written as a rational Bézier cubic.
- It has genus zero.
- It is *singular*, with exactly one double point.

An implicit cubic curve that passes through $\mathbf{P_0}$ tangent to $\mathbf{P_0P_1}$ and $\mathbf{P_2}$ tangent to $\mathbf{P_1P_2}$ has an equation of the form (Sederberg, 1984)

$$F = as^2u + bsu^2 - cst^2 - dt^2u + estu - ft^3 = 0 \tag{1}$$

The curvatures of F at $\mathbf{P_0}$ and $\mathbf{P_2}$ are given by

$$\kappa_0 = 4\frac{c}{a}\frac{A}{g_0^3} \qquad \kappa_2 = 4\frac{d}{b}\frac{A}{g_2^3}$$

where A is the area of the triangle, g_0 is the length of the side $\mathbf{P_0P_1}$, and g_2 is the length of the side $\mathbf{P_1P_2}$ (Paluszny and Patterson). In this paper we refer to the ratios

$$k_0 = \frac{c}{a} \qquad \text{and} \qquad k_2 = \frac{d}{b}$$

as curvatures, although in reality they are only proportional to the true curvatures.

Section 3. Formulas for the Bézier Cubic

In this section we find the formulas for the Bézier control points and weights in terms of the curvatures and the singular point. Because we are working in a triangle, the control points can be assigned barycentric coordinates $\mathbf{b_0}(1,0,0)$, $\mathbf{b_1}(X, 1-X, 0)$, $\mathbf{b_2}(0, 1-Y, Y)$ and $\mathbf{b_3}(0,0,1)$. We refer to X and Y as the *control point ratios* of $\mathbf{b_1}$ and $\mathbf{b_2}$.

Theorem 1. The formulas for the control point ratios X and Y and the normalized weights w_1, w_2 (with $w_0 = w_3 = 1$) in terms of k_0, k_2 and the singular point $\bar{\mathbf{B}}(\bar{s}, \bar{t}, \bar{u})$ are

$$X = \frac{-k_2\bar{t}(k_0\bar{t}^2 - \bar{s}\bar{u})}{\bar{u}(k_2\bar{t}^2 - \bar{s}\bar{u}) - k_2\bar{t}(k_0\bar{t}^2 - \bar{s}\bar{u})} \qquad Y = \frac{-k_0\bar{t}(k_2\bar{t}^2 - \bar{s}\bar{u})}{\bar{s}(k_0\bar{t}^2 - \bar{s}\bar{u}) - k_0\bar{t}(k_2\bar{t}^2 - \bar{s}\bar{u})}$$

$$w_1 = \frac{1}{3}\left(\frac{XY^2}{k_0^2k_2(X-1)^4(Y-1)^2}\right)^{1/3} \qquad w_2 = \frac{1}{3}\left(\frac{X^2Y}{k_0k_2^2(X-1)^2(Y-1)^4}\right)^{1/3}$$

Proof: There is a standard method for parametrizing the cubic F with singular point $\bar{\mathbf{B}}$ (Brieskorn and Knorrer, 1986). Using T as a parameter, the family of lines through $\bar{\mathbf{B}}$ is given by

$$L(T) = T(\bar{t}s - \bar{s}t) + (1-T)(\bar{u}t - \bar{t}u) = 0\ .$$

Using resultants to eliminate first s and then u between F and $L(T)$, we obtain formulas for u/t and s/t. When these are made homogeneous, we have a parametrization of F in terms of T:

$$\begin{aligned}
s =& (-\bar{s}\bar{u}^3 + k_2\bar{t}^2\bar{u}^2 + k_2\bar{s}\bar{t}^2\bar{u} - k_0k_2\bar{t}^4)T^3 + (3\bar{s}\bar{u}^3 - 3k_2\bar{t}^2\bar{u}^2 - 2k_2\bar{s}\bar{t}^2\bar{u} + 2k_0k_2\bar{t}^4)T^2 \\
&+ (-3\bar{s}\bar{u}^3 + 3k_2\bar{t}^2\bar{u}^2 + k_2\bar{s}\bar{t}^2\bar{u} - k_0k_2\bar{t}^4)T - \bar{u}^2(k_2\bar{t}^2 - \bar{s}\bar{u}) \\
t =& \bar{t}(\bar{s}^2\bar{u} - k_0\bar{s}\bar{t}^2 + k_2\bar{t}^2\bar{u} - \bar{s}\bar{u}^2)T^3 - \bar{t}(\bar{s}^2\bar{u} - k_0\bar{s}\bar{t}^2 + 2k_2\bar{t}^2\bar{u} - 2\bar{s}\bar{u}^2)T^2 \\
&+ \bar{t}\bar{u}(k_2\bar{t}^2 - \bar{s}\bar{u})T \\
u =& (\bar{s}^3\bar{u} - k_0\bar{s}^2\bar{t}^2 - k_0\bar{s}\bar{t}^2\bar{u} + k_0k_2\bar{t}^4)T^3 - k_0\bar{t}^2(k_2\bar{t}^2 - \bar{s}\bar{u})T^2
\end{aligned}$$

On the other hand, the standard Bézier parametrization of the same curve is

$$\begin{aligned}\mathbf{B}(T) =&(1-T)^3 w_0(1,0,0) + 3T(1-T)^2 w_1(X, 1-X, 0) + 3T^2(1-T)w_2(0, 1-Y, Y)\\ &+ T^3 w_3(0,0,1)\end{aligned}$$

From this

$$\begin{aligned}s &= (-w_0 + 3w_1X)T^3 + (3w_0 - 6w_1X)T^2 + (-3w_0 + 3w_1X)T + w_0\\ t &= (3w_1(1-X) - 3w_2(1-Y))T^3 + (-6w_1(1-X) + 3w_2(1-Y))T^2 + 3w_1(1-X)T\\ u &= (-3w_2Y + w_3)T^3 + 3w_2YT^2\end{aligned}$$

By equating coefficients of T, we obtain the formulas given for X and Y, as well as the following formulas for the weights:

$$\begin{aligned}w_0 &= -\bar{u}^2(k_2\bar{t}^2 - \bar{s}\bar{u})\\ w_1 &= \frac{\bar{t}(\bar{u}(k_2\bar{t}^2 - \bar{s}\bar{u}) - k_2\bar{t}(k_0\bar{t}^2 - \bar{s}\bar{u}))}{3}\\ w_2 &= \frac{\bar{t}(\bar{s}(k_0\bar{t}^2 - \bar{s}\bar{u}) - k_0\bar{t}(k_2\bar{t}^2 - \bar{s}\bar{u}))}{3}\\ w_3 &= -\bar{s}^2(k_0\bar{t}^2 - \bar{s}\bar{u})\end{aligned}$$

The formulas stated in the theorem for the weights are gotten by normalizing them so that $w_0 = w_3 = 1$ (Patterson, 1985) or (Farin, 1993) and making some substitutions. ∎

We will see in Section 7 that $0 \le X, Y \le 1$ for our interpolating cubics. The formulas in Theorem 1 then show that w_1 and w_2 are nonnegative.

4. The Singular Point Curve

With respect to the triangle $\mathbf{P_0P_1P_2}$ and curvatures k_0 and k_2, equation (1) takes the form

$$F = a(s^2u - k_0st^2) + b(su^2 - k_2t^2u) + estu - ft^3 = 0.$$

Modeling with these generally nonsingular curves is studied in (Paluszny and Patterson).

Here we are interested in *singular* curves that have the form of F. The condition that F should have a specific double point $\bar{\mathbf{B}}(\bar{s}, \bar{t}, \bar{u})$ is a linear condition in the coefficients a, b, e and f. Solving

$$F_s(\bar{s}, \bar{t}, \bar{u}) = F_t(\bar{s}, \bar{t}, \bar{u}) = F_u(\bar{s}, \bar{t}, \bar{u}) = 0$$

we obtain

$$
\begin{aligned}
a &= \bar{t}^3\bar{u}(k_2\bar{t}^2 - \bar{s}\bar{u}) \\
b &= \bar{s}\bar{t}^3(k_0\bar{t}^2 - \bar{s}\bar{u}) \\
e &= 3\bar{s}^2\bar{t}^2\bar{u}^2 - 2(k_0 + k_2)\bar{s}\bar{t}^2\bar{u} + k_0k_2\bar{t}^6 \\
f &= \bar{s}\bar{u}(\bar{s}^2\bar{u}^2 - k_0k_2\bar{t}^4)
\end{aligned} \tag{2}
$$

We see there is a unique cubic in our family with singular point $\bar{\mathbf{B}}$:

$$
\begin{aligned}
F =& \bar{t}^3\bar{u}(k_2\bar{t}^2 - \bar{s}\bar{u})(s^2u - k_0st^2) + \bar{s}\bar{t}^3(k_0\bar{t}^2 - \bar{s}\bar{u})(su^2 - k_2t^2u) + \\
& (3\bar{s}^2\bar{t}^2\bar{u}^2 - 2(k_0 + k_2)\bar{s}\bar{t}^4\bar{u} + k_0k_2\bar{t}^6)stu - \bar{s}\bar{u}(\bar{s}^2\bar{u}^2 - k_0k_2\bar{t}^4)t^3 = 0
\end{aligned}
$$

Another linear condition that can be imposed is that the cubic should interpolate a given point $\mathbf{B_0}(s_0, t_0, u_0)$. Now the possible singular points are restricted to lie on the sextic curve

$$
\begin{aligned}
S =& t^3u(k_2t^2 - su)(s_0^2u_0 - k_0s_0t_0^2) + st^3(k_0t^2 - su)(s_0u_0^2 - k_2t_0^2u_0) + \\
& (3s^2t^2u^2 - 2(k_0 + k_2)st^4u + k_0k_2t^6)s_0t_0u_0 - su(s^2u^2 - k_0k_2t^4)t_0^3 = 0
\end{aligned}
$$

After rearrangement, the equation of this curve, the *singular point* curve, becomes

$$
\begin{aligned}
S =& k_0k_2s_0t_0u_0t^6 + (k_0s_0u_0^2 - k_0k_2t_0^2u_0)st^5 + (k_2s_0^2u_0 - k_0k_2s_0t_0^2)t^5u - \\
& (s_0u_0^2 - k_2t_0^2u_0)s^2t^3u - (s_0^2u_0 - k_0s_0t_0^2)st^3u^2 - t_0^3s^3u^3 - \\
& (k_0k_2t_0^3 - 2(k_0 + k_2)s_0t_0u_0)st^4u + 3s_0t_0u_0s^2t^2u^2 = 0.
\end{aligned}
$$

Theorem 2. The singular point curve is an elliptic curve of degree six that has triple points at $\mathbf{P_0}$ and $\mathbf{P_2}$, where it is tangent to $u = 0$ and $s = 0$, respectively, and has a double point at $\mathbf{B_0}$.

Proof: All of the first partial derivatives of S vanish at $\mathbf{B_0}$ but not all the second derivatives do, so $\mathbf{B_0}$ is a double point. Similarly all the first and second partials of S vanish at $\mathbf{P_0}$ and $\mathbf{P_2}$ but not all the third do, so these are triple points. To find the tangent line(s) at the triple point $\mathbf{P_0}(1, 0, 0)$, we set $s = 1$ in S. The terms of lowest degree are the product of the expressions for the tangent lines at $\mathbf{P_0}$. In this case the lowest degree terms simplify to u^3, so $u = 0$ is the only tangent line.

By resolving the singularity at $\mathbf{P_0}$ (or $\mathbf{P_2}$) as in (Abhyankar, 1990), we find the triple point $\mathbf{P_0}$ has an "infinitely near" double point, or double point in its first neighborhood. Thus it counts as $\binom{3}{2} + \binom{2}{2} = 4$ double points. Together, $\mathbf{P_0}$, $\mathbf{P_2}$ and $\mathbf{B_0}$ account for 9 double points. A sextic can have at most 10 double points. The difference $10 - 9 = 1$ is the genus and so we have an elliptic curve. Another proof of this fact will follow from the construction in Section 6 of a birational transformation between the singular point curve and a quartic with two double points. ∎

The cubic F and the singular point curve S are intimately related to each other, arising as they do from the same equation, but with the set of variables that is fixed interchanged with the set that is not fixed. Consequently a statement made about one curve has a corresponding statement about the other. For example,

a) Given a cubic with singular point $\bar{\mathbf{B}}$, interpolating a point $\mathbf{B_0}$, the singular point curve interpolates $\bar{\mathbf{B}}$ and has singular point $\mathbf{B_0}$.

Corresponds to

a′) Given a singular point curve with singular point $\mathbf{B_0}$, interpolating a point $\bar{\mathbf{B}}$, the cubic interpolates $\mathbf{B_0}$ and has singular point $\bar{\mathbf{B}}$.

A broader such pair of statements is:

b) Given a cubic F with singular point $\bar{\mathbf{B}}$, each of its points $\mathbf{B_0}$ is the singular point of a singular point curve that has triple points at $\mathbf{P_0}$ and $\mathbf{P_2}$ and interpolates $\bar{\mathbf{B}}$.

b′) Given a singular point curve S with singular point $\mathbf{B_0}$, each of its points $\bar{\mathbf{B}}$ is the singular point of a cubic that has contact given by k_0 and k_2 at $\mathbf{P_0}$ and $\mathbf{P_2}$ and interpolates $\mathbf{B_0}$.

Statement (b′) is the foundation of this paper. We must investigate which segments of S correspond to cubics with a geometrically reasonable arc inside the triangle. Then we need to parametrize these arcs of S so that we can easily move $\bar{\mathbf{B}}$ along them. The formulas derived in Section 3 are then used to compute the control points and weights of the cubic that is determined.

Section 5. Useful Arcs of the Singular Point Curve

Not all of the cubics with singular point on S are acceptable for modeling purposes.

A sampling of unacceptable curves is given in Figure 2.

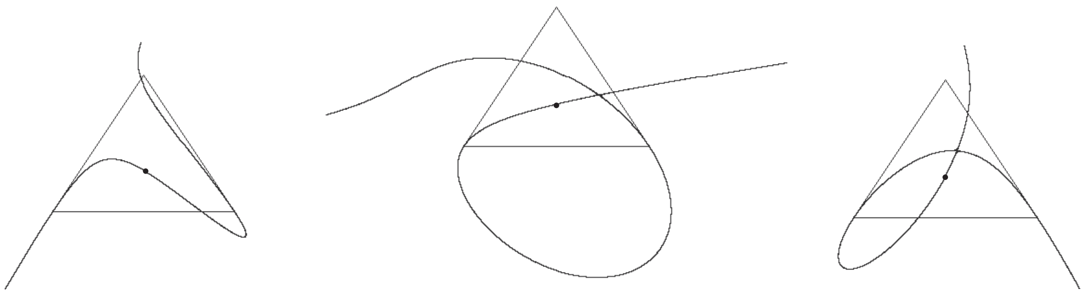

Figure 2. Unacceptable curves that satisfy the contact conditions.

For the purposes of this paper, we define a cubic to be *good* if it has a convex arc inside the triangle from $\mathbf{P_0}$ through $\mathbf{B_0}$ to $\mathbf{P_2}$ and if its singular point lies outside the triangle.

The curvatures k_0 and k_2 assigned at $\mathbf{P_0}$ and $\mathbf{P_2}$ determine two conics:

$$C_0 : k_0 t^2 - su = 0$$

$$C_2 : k_2 t^2 - su = 0$$

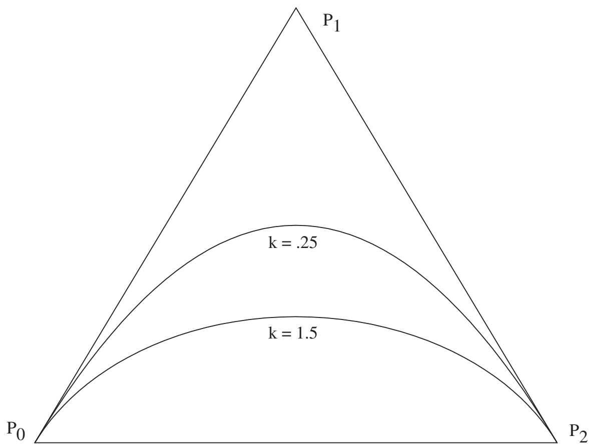

Figure 3. Conics determined by curvatures.

These conics help delineate the regions in which $\mathbf{B_0}$ and $\bar{\mathbf{B}}$ can lie in order for the cubic F to be good. The larger the value of k_0 or k_2, the closer is the arc of the conic C_0 or C_2 in the triangle to the segment $\mathbf{P_0P_2}$; see Figure 3.

The following theorem proves that there are two arcs of S that determine good cubics. In addition we will describe another arc of S, found experimentally, that determines cubics that do have a convex arc inside the triangle from $\mathbf{P_0}$ through $\mathbf{B_0}$ to $\mathbf{P_2}$, but that later reenter the triangle to cross the first arc. Although they are not good in our technical sense, these cubics are also acceptable for modeling since only the arc from $\mathbf{P_0}$ to $\mathbf{P_2}$ is used.

Theorem 3. If the interpolation point $\mathbf{B_0}$ lies in the interior of at least one of the two conics C_0 or C_2 then the singular point curve S has two arcs that determine good cubics. If $\mathbf{B_0}$ lies outside both conics, there is no solution to the interpolation problem.

The theorem is proved Section 7. Here we illustrate where the arcs lie on S and what types of cubic we obtain. The description depends on whether $\mathbf{B_0}$ lies inside both conics, or lies inside one and outside the other.

Figure 4 shows an example of the singular point curve when $\mathbf{B_0}$ lies inside both conics; this is detected by checking that both $q_0 = k_0 t_0^2 - s_0 u_0 = C_0(\mathbf{B_0})$ and $q_2 = k_2 t_0^2 - s_0 u_0 = C_2(\mathbf{B_0})$ are negative. The interpolation point $\mathbf{B_0}$ is an acnode of S. The indicated arc from $\mathbf{P_0}$ to $\mathbf{P_2}$ consists of the singular points of the *sloped* cubics. These curves have moderate-sized weights and are very useful for modeling. The sloped curve with singular point $\mathbf{A}$ is illustrated in Figure 5.

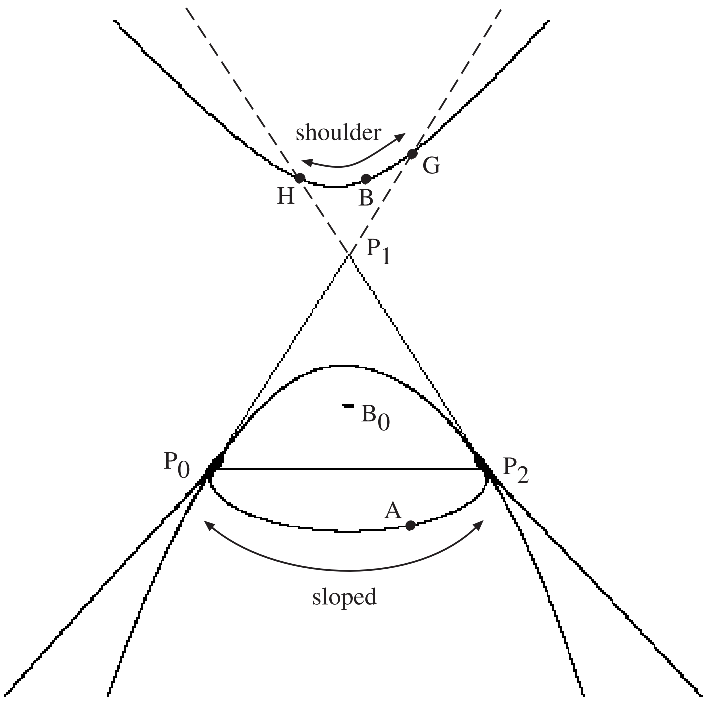

Figure 4. The singular point curve for $B_0(.35, .3, .35)$, $k_0 = 0.25$, $k_2 = 0.3$.

The other arc between the points $\mathbf{G}[k_2 s_0 t_0, q_2, 0]$ and $\mathbf{H}[0, q_0, k_0 t_0 u_0]$ on the lines $u = 0$ and $s = 0$ consists of the singular points of the *shoulder* cubics. These curves have fairly high weights. The shoulder curve with singular point $\mathbf{B}$ is also illustrated in Figure 5.

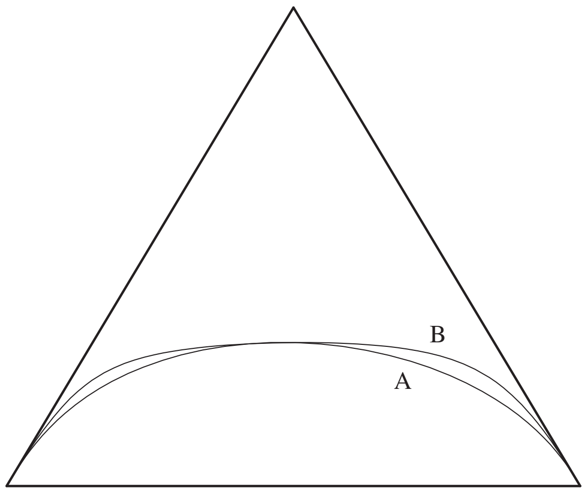

Figure 5. Examples of sloped (A) and shoulder (B) cubics.

Figure 6 shows an example of the singular point curve when $\mathbf{B_0}$ lies between the two conics; in this example the curvature k_0 is larger than k_2. The loop based at $\mathbf{P_0}$ now determines the sloped cubics. The arc that corresponds to shoulder cubics has stretched all the way to $\mathbf{P_2}$.

In addition to the arcs claimed in the Theorem, there is an additional arc that was discovered experimentally that gives useful cubics. It exists only when $\mathbf{B_0}$ lies between the two conics. The arc itself lies inside the triangle, so the cubic has its double point inside the triangle. The third arc corresponds to the *elbow* curves, which naturally extend the family of sloped curves. The transition from sloped curve to elbow curve is illustrated in Figure 7. All three types are illustrated in Figure 8, but this time they are graphed as implicit curves rather than Bézier curves as in Figure 5. Note the sloped curve A has an acnode at the point $\mathbf{A}$, the elbow curve B has a crunode inside the triangle at the point $\mathbf{B}$, while the shoulder curve C has a crunode at the point $\mathbf{C}$.

Careful inspection along the entire length of the singular point curve turned up no other acceptable families of cubics.

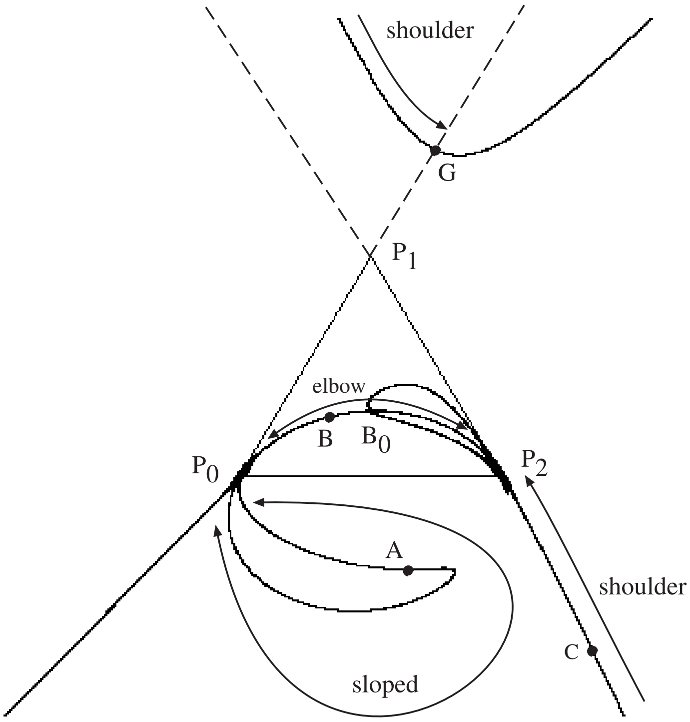

Figure 6. The singular point curve for $B_0(.35, .3, .35)$, $k_0 = 1.5$, $k_2 = 0.3$.

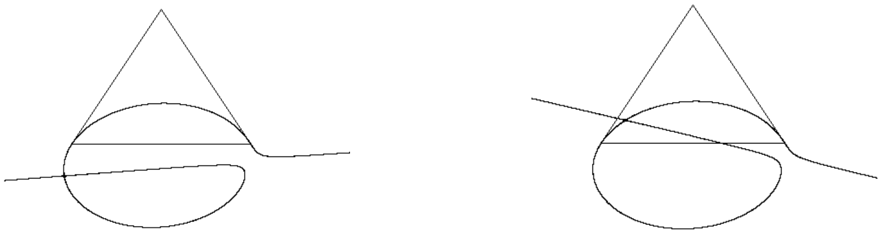

Figure 7. The transition from sloped curve to elbow curve.

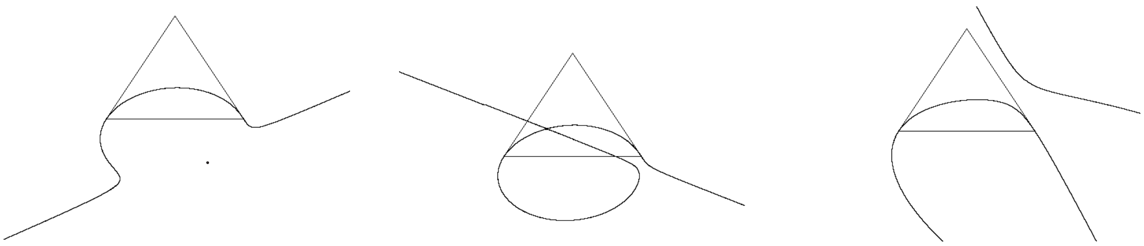

Figure 8. Examples of sloped (A), elbow (B) and shoulder (C) cubics.

6. The Quartic Guide Curve

In this Section we show that the singular point curve S is birationally equivalent to a curve Q of degree four called the *guide* curve. (The existence of this birational transformation constitutes an alternative proof that S is elliptic.) The guide curve is easier to work with because it has lower degree and is easy to parametrize. It will also be used in the proof of Theorem 3.

Theorem 4. The singular point curve S is birationally equivalent to a curve Q of degree four. (See Figures 9 and 10.) The guide curve is an elliptic curve with double points at $\mathbf{P_0'}(1,0,0)$ and $\mathbf{P_2'}(0,0,1)$. Its equation is

$$Q = -t_0^4 y^4 + s_0 t_0^3 x y^3 + t_0^3 u_0 y^3 z + t_0 u_0 q_2 x y z^2 + s_0 t_0 q_0 x^2 y z + q_0 q_2 x^2 z^2 \ , \tag{3}$$

where $q_0 = k_0 t_0^2 - s_0 u_0$ and $q_2 = k_2 t_0^2 - s_0 u_0$.

Proof: The following quadratic transformation Φ transforms (s,t,u)-space to (x,y,z)-space.

$$x = t_0 t (t_0 u - u_0 t)$$
$$y = t_0^2 s u - s_0 u_0 t^2$$
$$z = t_0 t (t_0 s - s_0 t)$$

The *fundamental points* where Φ is not defined are $\mathbf{P_0}$, $\mathbf{B_0}$ and $\mathbf{P_2}$. The *fundamental lines* are the three lines determined by these points. Away from these lines Φ is invertible, and its inverse transformation Φ^{-1} is given by

$$s = z(t_0 y - u_0 z)$$
$$t = t_0 x z$$
$$u = x(t_0 y - s_0 x)$$

The fundamental points of Φ^{-1} are $\mathbf{R_0}[t_0, s_0, 0]$, $\mathbf{P_1'}[0,1,0]$, and $\mathbf{R_2}[0, u_0, t_0]$. (For more information about quadratic transformations, see (Walker, 1950).)

When the substitution Φ^{-1} is made in the equation of the singular point curve we obtain the guide curve. As before, partial derivatives are used to check the statements about double points. ∎

It is possible to find a second quadratic transformation of the guide curve to a non-singular cubic. One uses the quadratic transformation with fundamental points $(1, 0, 0)$, $(0, 0, 1)$, and any third point of Q. While this reduces the degree again, making an arbitrary choice of the third point of Q breaks the symmetry inherent in the problem, so we work directly with the quartic guide curve.

The guide curves that are birationally equivalent to the singular point curves of Figures 4 and 6 are illustrated in Figures 9 and 10. The homogeneous coordinates of the transforms of $\mathbf{G}$ and $\mathbf{H}$ are $\mathbf{G}'[-q_2t_0, -q_2s_0, s_0^2t_0]$ and $\mathbf{H}'[t_0u_0^2, -q_0u_0, -q_0t_0]$. The arcs that correspond to the shoulder, sloped and elbow curves are labelled. The transforms of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are also marked.

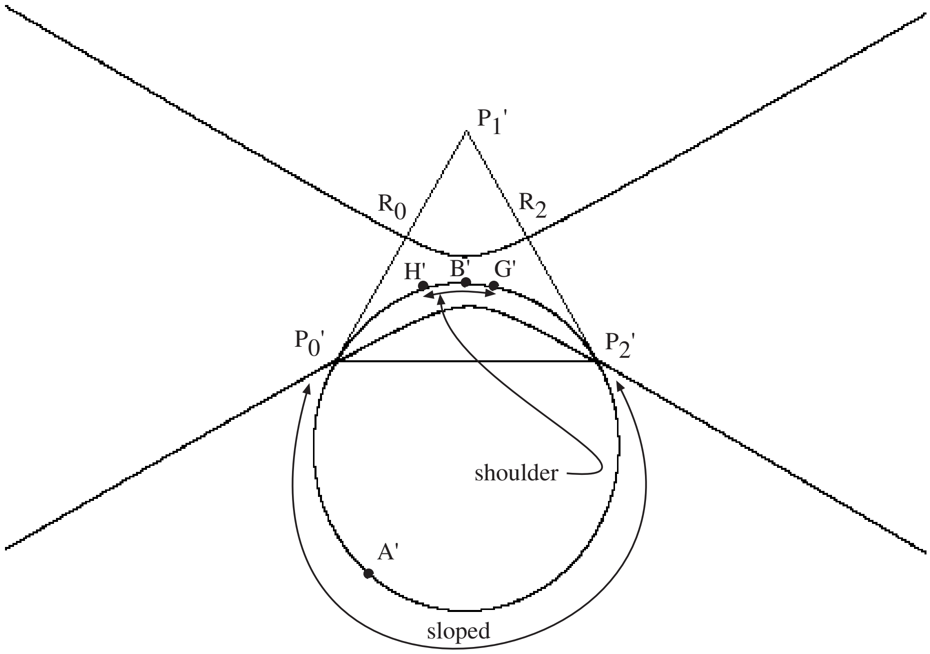

Figure 9. The guide curve corresponding to Figure 4

The guide curve can be made affine by setting $y = 1$ in (3); see Figures 12, 13 and 14. The affine guide curve is quadratic in each of x and z, so can be easily parametrized using one square root with either x or z as the parameter. By composing with Φ^{-1} we obtain a parametrization for the singular point curve. The parameter can be controlled by a slider, as mentioned in the introduction.

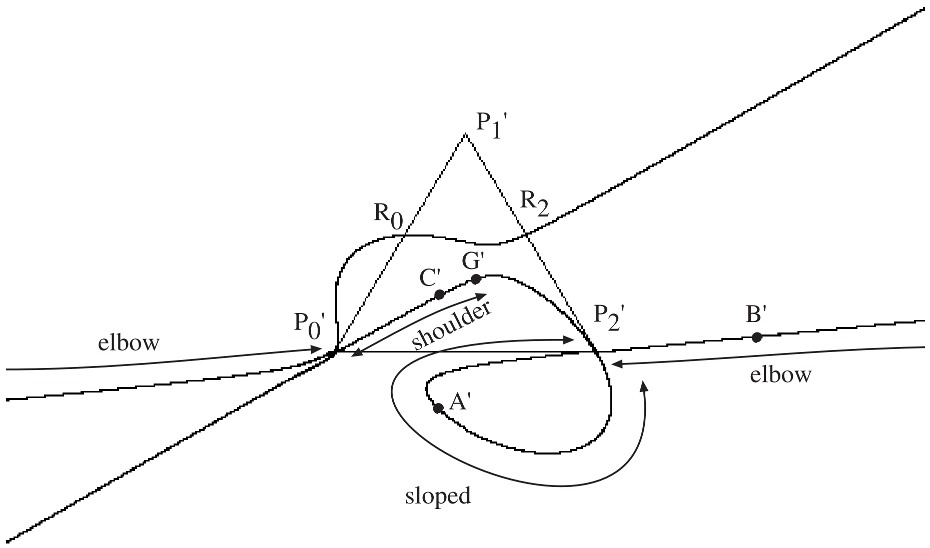

Figure 10. The guide curve corresponding to Figure 6

7. Proof of Theorem 3 and a Corollary

A cubic of the form (1) already crosses the line $\mathbf{P_0P_2}$ twice, so by Bezout's Theorem can cross it at most once more. For a good cubic this third crossing cannot be in the segment $\mathbf{P_0P_2}$; this means that the two coefficients a, b of F must have the same sign, or that $ab > 0$. We begin the proof by using this inequality to restrict the region of (s, t, u)-space in which $\bar{\mathbf{B}}$ must lie. This in turn restricts the region of (x, z)-space where we look for arcs of the affine guide curve.

Working first in (s, t, u)-space, the condition $ab > 0$, together with (2), says

$$\bar{s}\bar{u}(k_0\bar{t}^2 - \bar{s}\bar{u})(k_2\bar{t}^2 - \bar{s}\bar{u}) > 0 \tag{4}$$

from which it is easily seen that $\bar{s}\bar{u}$ must be positive, hence $\bar{s}$ and $\bar{u}$ have the same sign. Requiring $\bar{\mathbf{B}}$ to lie outside the triangle means $\bar{t}$ has the opposite sign. The shaded portions of Figure 11(a) are the regions of (s, t, u)-space to which $\bar{\mathbf{B}}$ is restricted.

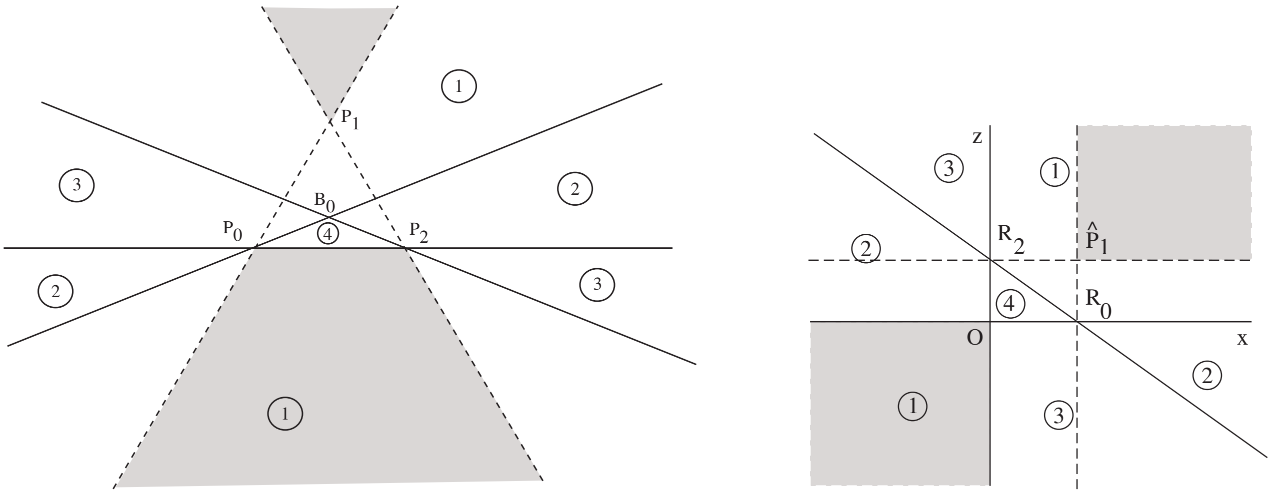

Figure 11. Corresponding Regions in (a) (s, t, u)-space, (b) (x, z)-space.

To determine the corresponding regions of (x, z)-space, we use the following easily determined facts about Φ:

- $\mathbf{P_1}$ transforms to $\hat{\mathbf{P}}_\mathbf{1}(\frac{t_0}{s_0}, \frac{t_0}{u_0})$.
- $s = 0$ transforms to $z = \frac{t_0}{u_0}$.
- $u = 0$ transforms to $x = \frac{t_0}{s_0}$.
- $t = 0$ (a fundamental line) transforms to $\mathbf{O}(0, 0)$.

Hence we can determine the images under Φ of the four regions 1,2,3,4 determined by the fundamental triangle $\mathbf{P_0 B_0 P_2}$. Corresponding regions are numbered in Figures 11(a) and 11(b) (ignore the shading and dotted lines). Region 1 is split into four subregions by the dotted lines. The shaded region in Figure 11(a) is bordered by $t = 0$, so corresponds to the shaded region in Figure 11(b) with vertex at $\mathbf{O}(0, 0)$.

In addition, we see from (4) that $C_0(\bar{\mathbf{B}})$ and $C_2(\bar{\mathbf{B}})$ must have the same sign. Thus $\bar{\mathbf{B}}$ must either be inside both conics or outside both conics.

The conics C_0 and C_2 transform via Φ to conics in (x, z)-space:

$$G_0 = q_0 xz + s_0 t_0 x + t_0 u_0 z - t_0^2$$

$$G_2 = q_2 xz + s_0 t_0 x + t_0 u_0 z - t_0^2$$

Thus we are looking for the arcs of the guide curve that lie in the shaded region of Figure 11(b) and either inside or outside both conics G_0, G_2.

The proof now splits into cases according to the signs and relative sizes of q_0 and q_2. The following facts are used in the proof.

(a) The graph of $Q(x, z) = 0$ crosses the axes only at $\mathbf{R_0}(\frac{t_0}{s_0}, 0)$ and $\mathbf{R_2}(0, \frac{t_0}{u_0})$.

(b) The curve $Q(x, z) = 0$ crosses the line $z = \frac{t_0}{u_0}$ exactly twice: simple intersections at $\mathbf{R_2}$ and at $(\frac{-t_0 u_0}{q_0}, \frac{t_0}{u_0})$.

(c) The curve $Q(x, z) = 0$ crosses the line $x = \frac{t_0}{s_0}$ exactly twice: simple intersections at $\mathbf{R_0}$ and at $(\frac{t_0}{s_0}, \frac{-s_0 t_0}{q_2})$.

(d) The curve $Q = 0$ intersects each $G_i = 0$ in exactly two finite points, $\mathbf{R_0}$ and $\mathbf{R_2}$.

(e) As a polynomial of degree two in z,

$$Q(x, z) = A(x)z^2 + B(x)z + C(x)$$

where

$$\begin{aligned} A(x) &= q_2 x(t_0 u_0 + q_0 x) \\ B(x) &= t_0(t_0^2 u_0 + s_0 q_0 x^2) \\ C(x) &= t_0^3(s_0 x - t_0) \end{aligned}$$

The guide curve Q can be solved for z in terms of its discriminant $\Delta(Q) = B(x)^2 - 4A(x)C(x)$:

$$\begin{aligned} z_1 &= \frac{-B - \sqrt{\Delta(Q)}}{2A} \\ z_2 &= \frac{-B + \sqrt{\Delta(Q)}}{2A} \end{aligned}$$

Case 1. Assume that $\mathbf{B_0}$ lies in the interior of both conics C_0 and C_2, and in particular that $q_2 < q_0 < 0$. Figure 12 shows the shaded region of the (x, z)-plane in which we are searching for arcs of the guide curve, and the guide curve itself which appears dotted. We will see that there are exactly two arcs of $z_1 = 0$ in the shaded region and no arcs of z_2.

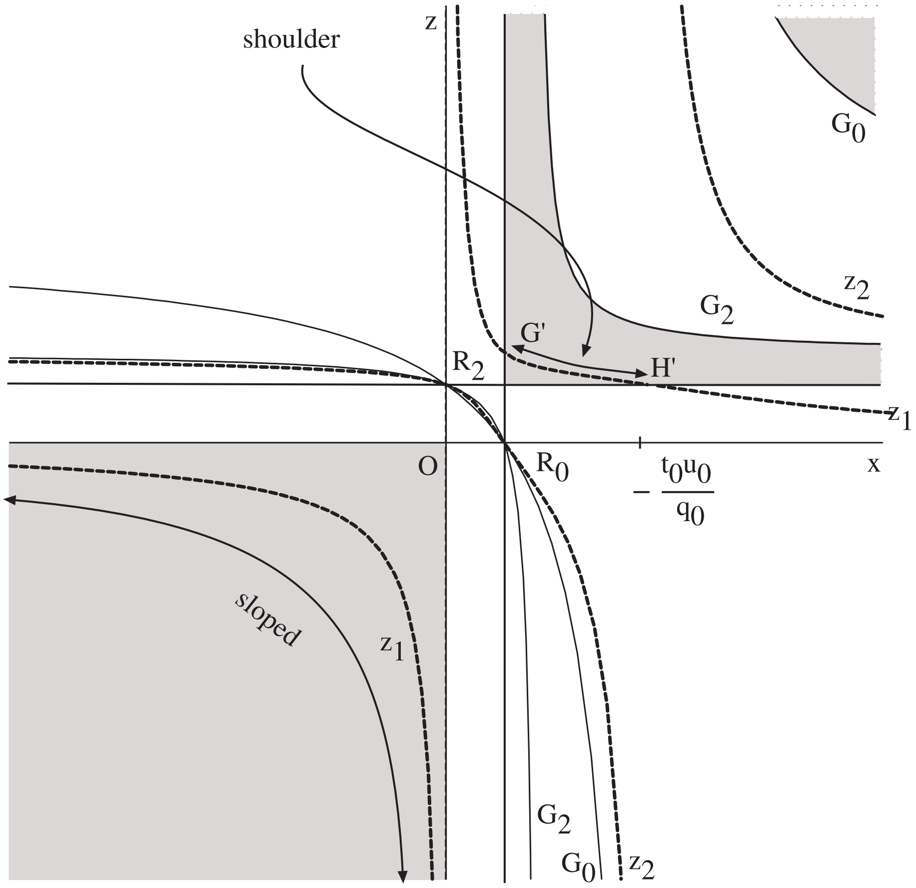

Figure 12. The affine guide curve (dotted) for $\mathbf{B_0}(.3, .4, .3), k_0 = .4, k_2 = .2$.

For $x < 0$, $-AC > 0$, so that $\sqrt{\Delta(Q)}$ is real. Because $\sqrt{\Delta(Q)} > |B(x)|$, $z_1 < 0$ but $z_2 > 0$. The graph of z_1 in the third quadrant therefore determines good cubics – the sloped cubics.

For $x \in (\frac{t_0}{s_0}, \frac{-t_0u_0}{q_0})$, again $-AC > 0$, so that $\sqrt{\Delta(Q)}$ is real, $z_1 > 0$ and $z_2 < 0$. We see immediately that the graph of z_2 does not enter the shaded region. For x in this interval we need to see that the graph of z_1 lies above $z = \frac{t_0}{u_0}$ and below the graph of G_2. Using $z_1(\frac{t_0}{s_0}) = -\frac{s_0t_0}{q_0} > \frac{t_0}{u_0}$ and $z_1(\frac{-t_0u_0}{q_0}) = \frac{t_0}{u_0}$ and assertion (b) we see that the graph of z_1 lies above $z = \frac{t_0}{u_0}$ in this interval. Furthermore since the vertical asymptote of G_2 occurs at $x = -\frac{t_0u_0}{q_2}$, which is inside the x-interval, and by (d) the graphs of Q and G_2 do not cross in the interval, the graph of z_1 lies below the graph of G_2. This arc of $Q = 0$ determines the shoulder cubics. It can not be extended past $x = \frac{-t_0u_0}{q_0}$ because the graph of z_1 has crossed the line $z = \frac{t_0}{u_0}$.

The proof in the case $q_0 < q_2 < 0$ is the same for $x < 0$. For $x \in (\frac{t_0}{s_0}, \frac{-t_0 u_0}{q_0})$ the graphs of G_0 and G_2 are interchanged and one shows that the graph of z_1 lies below the graph of G_0.

Case 2. Now assume that $\mathbf{B_0}$ lies between the two conics C_0 and C_2, and in particular that $q_2 < 0 < q_0$. Figure 13 shows the shaded region of the (x, z)-plane in which we are searching for arcs of the guide curve, and the dotted guide curve. Again we will see that there are two arcs of $z_1 = 0$ in the shaded region that correspond to the sloped and shoulder arcs. In addition, the arc that corresponds to the elbow cubics is labelled; it also is part of the graph of z_1.

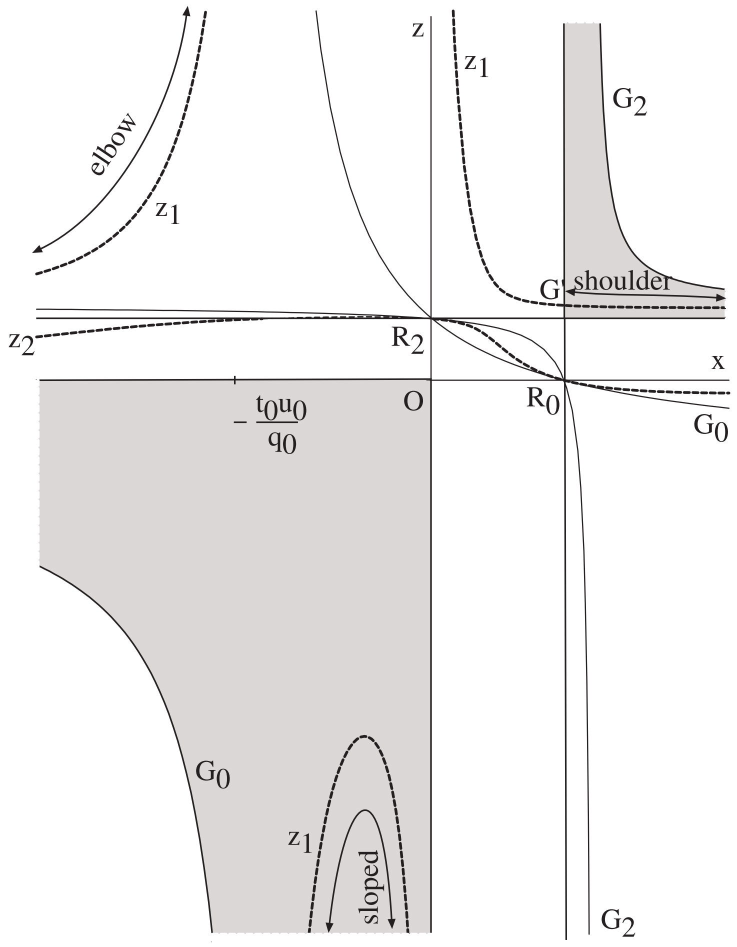

Figure 13. The affine guide curve (dotted) for $\mathbf{B_0}(.3, .4, .3), k_0 = 1, k_2 = .1$.

For $x \in (-\frac{t_0 u_0}{q_0}, 0)$, $\sqrt{\Delta(Q)}$ is real, $z_1 < 0$ and $z_2 > 0$. The graph of z_1 in this interval is the arc of $Q = 0$ that determines the sloped cubics. It cannot be extended for $x < -\frac{t_0 u_0}{q_0}$, for there both $z_1, z_2 > 0$ (although one of these arcs determines the elbow cubics).

For $x \in (\frac{t_0}{s_0}, \infty)$, again $-AC > 0$, so that $\sqrt{\Delta(Q)}$ is real, $z_1 > 0$ and $z_2 < 0$. Using $z_1(\frac{t_0}{s_0}) = -\frac{s_0 t_0}{q_0} > \frac{t_0}{u_0}$ and assertion (b) we see that the graph of z_1 lies above $z = \frac{t_0}{u_0}$ in this interval. And for the same reason as in case 1, the graph of z_1 lies below the graph of G_2. The graph of z_1 over this interval is the arc of $Q = 0$ that determines the shoulder cubics.

To conclude this case, we have to consider the situation $q_0 < 0 < q_2$. The proof is similar, except that now we solve $Q = 0$ for x in terms of z. The two intervals for z are $(-\frac{s_0 t_0}{q_2}, 0)$ and $(\frac{t_0}{u_0}, \infty)$.

Case 3. Finally assume that $\mathbf{B_0}$ does not lie inside either conic C_0 or C_2, and in particular that $0 < q_0 < q_2$. We will see there are no arcs of $Q = 0$ in any of the zones of the (x, z)-plane shaded in Figure 14.

We start by checking that the intersection of the guide curve with the shaded zone in the first quadrant is empty. After performing the substitution $x = r + \frac{t_0}{s_0}, z = w + \frac{t_0}{u_0}$ in equation (3) for $Q = 0$, we see that every sign in the resulting equation is positive. Hence there are no solutions for $r > 0$ and $w > 0$.

Next consider the third quadrant. The following two equations

$$(q_2 xz + t_0^2)G_0 - Q = -t_0^3 xz(k_2 - k_0)(t_0 - s_0 x)$$
$$(q_0 xz + t_0^2)G_2 - Q = t_0^3 xz(k_2 - k_0)(t_0 - u_0 z)$$

relate Q to the conics. For points in the third quadrant, the right hand side of the first is negative while that of the second is positive, while the coefficients of G_0 and G_2 are positive. Hence no points of $Q = 0$ satisfy $G_0 G_2 > 0$.

Now that we have obtained arcs of the affine guide curve on which $ab > 0$, we must check that this is sufficient to guarantee that the points determine good cubics. By construction the singular points of the cubics lie outside the triangle. Next we show that the cubic crosses into the triangle at no points other than $\mathbf{P_0}$ and $\mathbf{P_2}$.

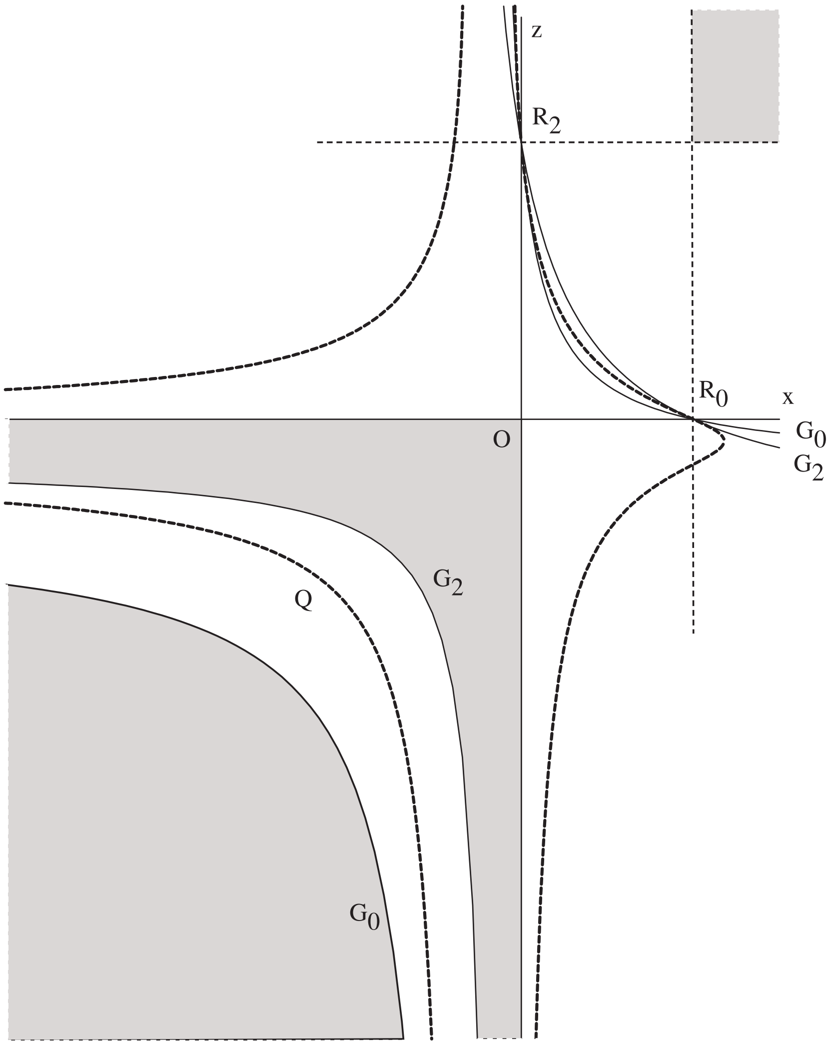

Figure 14. The affine guide curve (dotted) for $\mathbf{B_0}(.3,.4,.3), k_0 = 2, k_2 = 4$.

In order that the cubic does not cross between $\mathbf{P_0}$ and $\mathbf{P_1}$ we need c and f, therefore a and f to have the same sign, or $af > 0$. Similarly we need $bf > 0$. Thus in addition to (4) we want

$$-\bar{s}\bar{t}(k_2\bar{t}^2 - \bar{s}\bar{u})(k_0k_2\bar{t}^4 - \bar{s}^2\bar{u}^2) > 0$$

$$-\bar{t}\bar{u}(k_0\bar{t}^2 - \bar{s}\bar{u})(k_0k_2\bar{t}^4 - \bar{s}^2\bar{u}^2) > 0$$

Recall we are working at points where $\bar{s}, \bar{u}$ have the same sign opposite to that of $\bar{t}$. Note that when $k_0\bar{t}^2 - \bar{s}\bar{u}$ and $k_2\bar{t}^2 - \bar{s}\bar{u}$ have the same sign, $k_0k_2\bar{t}^4 - \bar{s}^2\bar{u}^2$ also has this sign. So inequality (4) implies both of these two inequalities. Hence a cubic entering the triangle at $\mathbf{P_0}$ must leave at $\mathbf{P_2}$, and being the only segment of the cubic in the triangle, must pass through $\mathbf{B_0}$ on the way. Finally the arc is convex by the convex hull property. ∎

Corollary. For good cubics, the control point ratios satisfy

$$0 \le X, Y \le 1$$

Proof: The proof uses the fact that

$$0 < \frac{a}{a+b} < 1 \quad \text{if and only if} \quad ab > 0$$

For X this is used with $a = -k_2 \bar{t} C_0(\bar{\mathbf{B}})$ and $b = \bar{u} C_2(\bar{\mathbf{B}})$. Then

$$ab = -k_2 \bar{t} \bar{u} C_0(\bar{\mathbf{B}}) C_2(\bar{\mathbf{B}})$$

is positive because $\bar{t}$ and $\bar{u}$ have opposite signs and $C_0(\bar{\mathbf{B}})$ and $C_2(\bar{\mathbf{B}})$ have the same sign.
∎

8. The Case of Curvature Zero.

In this section we study what happens when one or both of k_0 or k_2 is zero.

The shoulder curves tend to have fairly large weights anyway, but as the curvature k_0 tends to zero, the weight w_1 on that side tends to infinity. The limiting shoulder cubic when $k_0 = 0$ factors into the product of the line $u = 0$ and a conic. Similarly when $k_2 = 0$ the cubic factors into the product of the line $s = 0$ and a conic. When both k_0 and k_2 equal zero, the cubic factors as the product of three lines. Because of their tendency to have large weights and the fact that they are reducible when a curvature equals zero, the shoulder cubics are perhaps less useful for modeling than the sloped and elbow cubics. The sloped and elbow curves have more moderate weights and do not all become reducible if k_0 and/or k_2 equals zero. These curves can be used to introduce inflection points where needed in splines.

However, we must discuss three situations for sloped and elbow cubics when k_0 and/or k_2 is zero:

- one of k_0, k_2 is zero but the other is not, and $\mathbf{B_0}$ lies inside both conics,
- one of k_0, k_2 is zero but the other is not, and $\mathbf{B_0}$ is between the two conics,
- both of k_0, k_2 are zero, in which case $\mathbf{B_0}$ necessarily lies inside both (reducible) conics – anywhere in the triangle.

First if $k_2 = 0$, $k_0 \neq 0$ and $q_0 = k_0 t_0^2 - s_0 u_0 < 0$, $\mathbf{B_0}$ is inside both conics. There are sloped curves but no elbow curves. The guide curve factors

$$Q = (t_0 - u_0 z)(-t_0^3 + s_0 t_0^2 k_0 x^2 z + s_0 t_0^2 x + s_0 t_0 u_0 x z - s_0^2 u_0 x^2 z)$$

(see Figure 15). If we choose $x < 0$ we can easily compute z. The cubic that corresponds to the point $\mathbf{A}$ is the intermediate curve graphed in Figure 16.

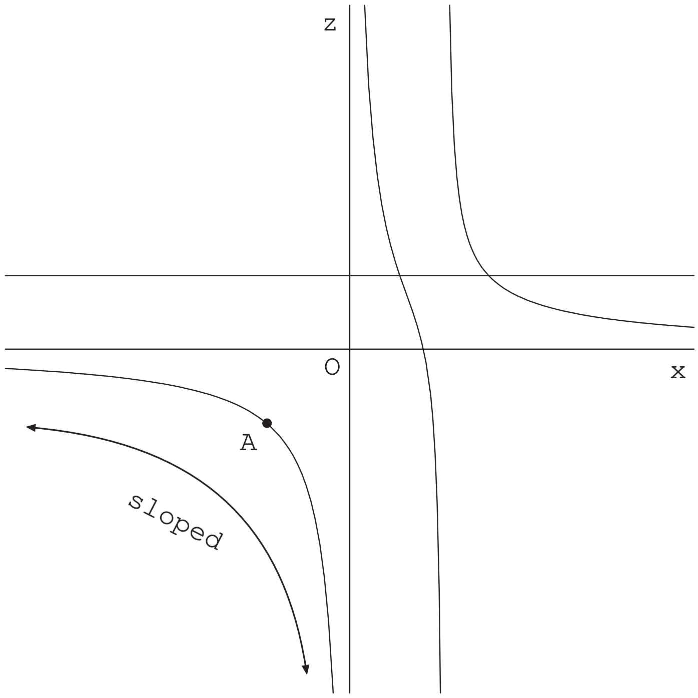

Figure 15. The affine guide curve for $B_0(.35, .3, .35)$, $k_0 = 0.3$, $k_2 = 0$.

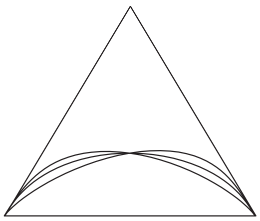

Figure 16. The range of cubics with curvature zero at P_2; $\mathbf{B_0}(.35, .3, .35)$, $k_0 = 0.3$, $k_2 = 0$.

If $k_2 = 0$ and $q_0 > 0$, $\mathbf{B_0}$ lies between the two conics. A typical graph of the guide curve appears in Figure 17. The intersection point $\mathbf{I}$ is

$$\left(\frac{t_0(-s_0 u_0 - t_0 \sqrt{s_0 u_0 k_0})}{s_0 q_0}, \frac{t_0}{u_0} \right)$$

To the left of **I** the corresponding cubics are reducible and are no longer useful solutions to our interpolation problem. Thus the parameter domain, for x in this case, must be restricted to $(t_0(-s_0u_0 - t_0\sqrt{s_0u_0k_0})/s_0q_0, 0)$.

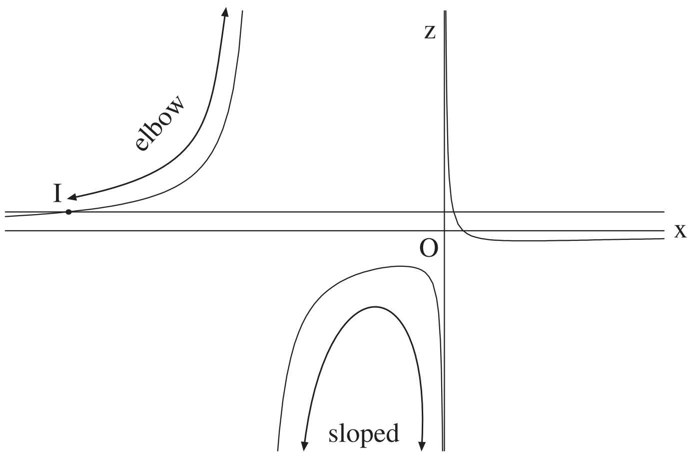

Figure 17. The affine guide curve for $\mathbf{B_0}(.35, .3, .35)$, $k_0 = 1.5$, $k_2 = 0$.

When both k_0 and k_2 equal zero, the guide curve factors

$$Q = (t_0 - u_0z)(t_0 - s_0x)(s_0u_0xz - t_0^2)$$

(see Figure 18). The cubic in Figure 19 that corresponds to **A** has zero curvature at both ends.

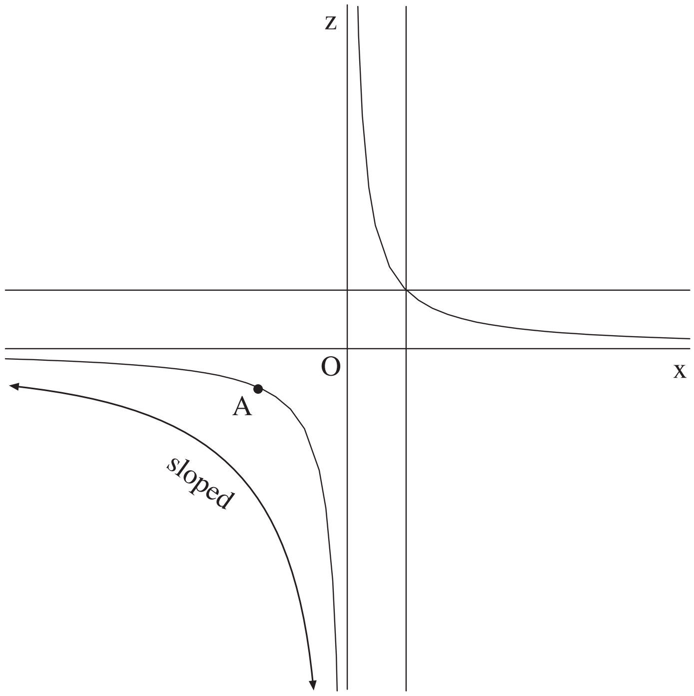

Figure 18. The affine guide curve for $B_0(.35, .3, .35)$, $k_0 = 0$, $k_2 = 0$.

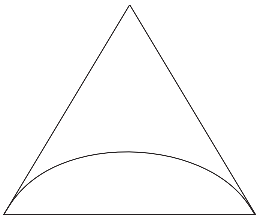

Figure 19. A cubic with curvature zero at both P_0 and P_2.

When either or both curvatures are zero, some modifications are needed in the formulas in Theorem 3. If $k_0 = 0$ then $Y = 0$, so the control point $\mathbf{b_2} = \mathbf{P_1}$. Similarly if $k_2 = 0$, $X = 0$, so that $\mathbf{b_1} = \mathbf{P_1}$. In either of these cases the formulas given for w_1 and w_2 fail, as there are zeros in the denominator. However the formulas for w_1 and w_2 can be thought of as depending on X/k_2 and Y/k_0. It is easy to find the limit of Y/k_0 as $k_0 \longrightarrow 0$, so this formula can be substituted instead to find the formulas to use in this case:

$$w_1 = \frac{1}{3}\left(\frac{X\bar{t}^2(k_2\bar{t}^2 - \bar{s}\bar{u})^2}{\bar{s}^4\bar{u}^2k_2(X-1)^4}\right)^{1/3} \qquad w_2 = \frac{1}{3}\left(\frac{X^2\bar{t}(k_2\bar{t}^2 - \bar{s}\bar{u})}{\bar{s}^2\bar{u}k_2^2(X-1)^2}\right)^{1/3}$$

If $k_2 = 0$ and $k_0 \neq 0$ the formulas are

$$w_1 = \frac{1}{3}\left(\frac{Y^2\bar{t}(k_0\bar{t}^2 - \bar{s}\bar{u})}{\bar{s}\bar{u}^2k_0^2(Y-1)^2}\right)^{1/3} \qquad w_2 = \frac{1}{3}\left(\frac{Y\bar{t}^2(k_0\bar{t}^2 - \bar{s}\bar{u})^2}{\bar{s}^2\bar{u}^4k_0(Y-1)^4}\right)^{1/3}$$

If both k_0 and k_2 equal zero,

$$w_1 = -\frac{1}{3}\left(\frac{\bar{t}^3}{\bar{s}^2\bar{u}}\right)^{1/3} \qquad w_2 = -\frac{1}{3}\left(\frac{\bar{t}^3}{\bar{s}\bar{u}^2}\right)^{1/3}$$

9. Programming Details and Summary

In this section we go through several other details that must be managed in a program to produce the sloped and elbow cubics.

It is reasonable to constrain the location of $\mathbf{B_0}$ to the line segment joining the midpoint of triangle side $\mathbf{P_0P_2}$ to the opposite vertex $\mathbf{P_1}$. Then moving $\mathbf{B_0}$ is like adjusting the weight for a quadratic Bézier curve. However, $\mathbf{B_0}$ must not be permitted to go above the taller of the two conics. If a curvature is changed so that the taller conic moves down, when it reaches $\mathbf{B_0}$ it must push $\mathbf{B_0}$ down with it.

When $q_0 > 0$ the negative x-axis is divided by an asymptote at $x = -\frac{t_0 u_0}{q_0}$ into the domain for elbow curves and the domain for sloped curves; see Figure 13. Similarly when $q_2 > 0$ the negative z-axis is divided by an asymptote at $z = -\frac{s_0 t_0}{q_2}$. Since we must select x and compute z when $q_0 > 0$, and select z and compute x when $q_2 > 0$, it is reasonable to switch from selecting x when k_0 is greater than k_2, to selecting z when k_2 becomes greater than k_0. Of course when one of q_0, q_2 becomes positive it is also necessary to test for the asymptote after selecting x or z.

As we saw in the proof of Theorem 3, the good cubics for $k_0 > k_2$ were determined by points on the graph of z_1, never z_2. For $k_2 > k_0$, the correponding function x_1 is used. The following table summarizes the formulas and the domains for the three types of cubic:

		elbow	sloped	shoulder
$0 < k_0 < k_2$ $q_0 < 0, q_2 > 0$	Use x_1. Select z in	$\left(-\infty, -\frac{s_0 t_0}{q_2}\right)$	$\left(-\frac{s_0 t_0}{q_2}, 0\right)$	$\left(\frac{t_0}{u_0}, \infty\right)$
$0 = k_0 < k_2$ $q_0 < 0, q_2 > 0$	Use x_1. Select z in	$\left(\frac{t_0(-s_0 u_0 - t_0\sqrt{s_0 u_0 k_2})}{u_0 q_2}, -\frac{s_0 t_0}{q_2}\right)$	$\left(-\frac{s_0 t_0}{q_2}, 0\right)$	
$0 = k_0 \leq k_2$ $q_0 < 0, q_2 < 0$	Use x_1. Select z in		$(-\infty, 0)$	
$0 < k_0 < k_2$ $q_0 < 0, q_2 < 0$	Use x_1. Select z in		$(-\infty, 0)$	$\left(\frac{t_0}{u_0}, -\frac{s_0 t_0}{q_2}\right)$
$0 < k_2 < k_0$ $q_0 < 0, q_2 < 0$	Use z_1. Select x in		$(-\infty, 0)$	$\left(\frac{t_0}{s_0}, -\frac{t_0 u_0}{q_0}\right)$
$0 = k_2 \leq k_0$ $q_0 < 0, q_2 < 0$	Use z_1. Select x in		$(-\infty, 0)$	
$0 = k_2 < k_0$ $q_0 > 0, q_2 < 0$	Use z_1. Select x in	$\left(\frac{t_0(-s_0 u_0 - t_0\sqrt{s_0 u_0 k_0})}{s_0 q_0}, -\frac{t_0 u_0}{q_0}\right)$	$\left(-\frac{t_0 u_0}{q_0}, 0\right)$	
$0 < k_2 < k_0$ $q_0 > 0, q_2 < 0$	Use z_1. Select x in	$\left(-\infty, -\frac{t_0 u_0}{q_0}\right)$	$\left(-\frac{t_0 u_0}{q_0}, 0\right)$	$\left(\frac{t_0}{s_0}, \infty\right)$

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