The Singular Point of an Algebraic Cubic

Marco Paluszny, Richard Patterson and Francisco Tovar

Abstract. Given a singular algebraic cubic, tangent at two points to prescribed lines, formulas are given for the coordinates of its singular point. The coordinates of the singular point are expressed as rational functions of degree at most five in the coefficients of the cubic.

§1. Introduction

Singular algebraic cubics and rational Bézier curves of degree three are the same family of curves, which is widely used in Computer Aided Geometric Design (CAGD).

For curves that have already been parametrized, such as Bézier curves, the problem of determining the singular point(s) has been of interest in CAGD. The problem was studied for polynomial curves by Su Buchin and Dingyuan [9]. Stone and De Rose [8] did a geometric analysis for polynomial cubics. For rational cubics a solution is given in Patterson [3], another geometric analysis was provided by Pottmann and DeRose [4]. The problem of locating cusps in rational curves is discussed by Manocha and Canny in [1]. The problem of approximating double points was solved with an interactive procedure by Sederberg in [7].

But all of the papers mentioned above for finding double points require a parametrization to be given first, and yet given a singular algebraic curve, the usual method for finding a parametrization requires the double point to be known in advance.

The problem of determining the singularity of an algebraic curve of degree n, with rational coefficients, was also studied by Sakkalis and Farouki in [5]. Their method consists in isolating the coordinates of singular points in intervals and approximating them numerically. One of their formulas, in the case of cubics, can be extended to yield formulas similar to the ones given here, but of higher degree.

The goal of this paper is to provide explicit formulas for the coordinates of the singular point of a given implicit singular cubic in terms of its coefficients.

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§2. Expression of the Cubic

A point $P(x, y, z)$ is a singular point of the curve $F(x, y, z) = 0$ if $F_x(P) = F_y(P) = F_z(P) = 0$, where F_x, F_y and F_z denote the partial derivatives of F. Otherwise, the point P is called regular. Every cubic is known to have at most one singular point. At a regular point P, the tangent line is given by $F_x(P)x + F_y(P)y + F_z(P)z = 0$.

Suppose that $P_0(x_0, y_0, z_0)$ and $P_2(x_2, y_2, z_2)$ are two regular points of $F = 0$ and $P_1(x_1, y_1, z_1)$ is the intersection point of the tangent lines at P_0 and P_2.

Let (s, t, u) be the new system of homogeneous coordinates given by

$$(x, y, z) = (s, t, u)\begin{pmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

We then have barycentric coordinates if we also require that $s + t + u = 1$, but for our purposes this is only necessary in Section 4. In the new coordinate system the expression of the cubic is

$$F(s, t, u) = as^2u + bsu^2 - cst^2 - dt^2u + estu - ft^3. \tag{2.1}$$

Here the minus signs are incorporated in some coefficients to make later formulas simpler.

If $a = 0$ in (2.1), then P_0 is a singularity. Likewise, if $b = 0$, then P_2 is a singularity. Thus the problem of finding the singular point in these cases is solved, and consequently in the rest of the paper we make the standing assumption that a and b are different from zero. With this assumption, a singular point of $F = 0$ can not lie on any of the lines $s = 0$, $t = 0$, or $u = 0$. This is because each of these lines already has two points in common with the curve, so if one of the lines also passed through a singular point it would then have four-point contact with the curve, which is impossible by Bezout's theorem (see [10] p. 59).

§3. Singular Point of the Cubic

In this section we will see how to determine if a cubic has a singular point, and find formulas for the singular point when there is one. The following formula in the coefficients of F can be used to determine singularity.

Theorem 3.1. *The cubic $F(s, t, u) = 0$ given by equation (2.1) is singular or reducible if and only if $\Delta(a, b, c, d, e, f) = 0$, where*

$$\begin{aligned} \Delta = &- 16a^2cd^3ef - 16a^2c^2d^4 - 36ab^2cef^3 + 16b^3c^3f^2 \\ &+ 16a^3d^3f^2 - 8bc^3d^2e^2 - 8bc^2de^3f - 24ab^2c^2df^2 \\ &- 36a^2bdef^3 + 8a^2d^2e^2f^2 - 8ac^2d^3e^2 + 27a^2b^2f^4 \\ &- 24a^2bcd^2f^2 + 32abc^3d^3 + 64abc^2d^2ef + 46abcde^2f^2 \\ &- c^2d^2e^4 - cde^5f + cbe^4f^2 - 16b^2c^3def - 8acd^2e^3f \\ &- 16b^2d^2c^4 + ade^4f^2 - abe^3f^3 + 8b^2c^2e^2f^2. \end{aligned} \tag{3.1}$$

Proof: See [6] pages 189-192 or the discussion in [2]. ■

The following lemma gives a sufficient condition for the cubic F to be reducible.

Lemma 3.2. *If the coefficients c, d and f of the cubic F of (2.1) are different from zero and satisfy the condition $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$, then the cubic is reducible.*

Proof: Let α be the ratio in the Lemma. Substituting $a = c\alpha$, $b = d\alpha$ and $e = f\alpha$ we find $F = (\alpha su - t^2)(cs + ft + du)$. ■

Theorem 3.3. *If the cubic $F(s,t,u) = 0$ given by (2.1) is singular and irreducible, then the homogeneous coordinates of the singular point $(\overline{s}, 1, \overline{u})$ are given by the following formulas:*

1) If S_1 and U_1 are not zero,

$$\overline{s} = \frac{G_1(a,b,c,d,e,f)}{S_1(a,b,c,d,e,f)} \tag{3.2}$$

$$\overline{u} = \frac{G_1(a,b,c,d,e,f)}{U_1(a,b,c,d,e,f)} \tag{3.3}$$

2) If $S_1 = 0$ or $U_1 = 0$,

$$\overline{s} = \frac{G_2(a,b,c,d,e,f)}{S_2(a,b,c,d,e,f)} \tag{3.4}$$

$$\overline{u} = \frac{G_2(a,b,c,d,e,f)}{U_2(a,b,c,d,e,f)} \tag{3.5}$$

The explicit expressions of G_1, S_1, U_1, G_2, S_2 and U_2 are

$$\begin{aligned} G_1(a,b,c,d,e,f) &= 7cde^2f - 8adef^2 + 8c^2d^2e - 4acd^2f \\ &\quad - 8bcef^2 + 9abf^3 - 4bc^2df && (3.6) \\ S_1(a,b,c,d,e,f) &= -12acdfe + 12a^2df^2 - 8ac^2d^2 + ce^3f \\ &\quad - ae^2f^2 + 2c^2de^2 + 4bc^2ef - 6abcf^2 + 8bc^3d && (3.7) \\ U_1(a,b,c,d,e,f) &= S_1(b,a,d,c,e,f) \\ G_2(a,b,c,d,e,f) &= f(4cd - ef) \\ S_2(a,b,c,d,e,f) &= 2(3af^2 - 2cef - 4c^2d) && (3.8) \\ U_2(a,b,c,d,e,f) &= S_2(b,a,d,c,e,f) \end{aligned}$$

The proof uses the following

Lemma 3.4.
1) If the cubic F is irreducible and $S_1 = 0$ or $U_1 = 0$, then $G_1 = 0$.
2) If $G_1 = 0$ and either $S_2 = 0$ or $U_2 = 0$, then F is reducible.

The consequence of this lemma is that the formulas given in (3.2), (3.3), (3.4) and (3.5) cover all the possible situations. The proof is deferred until after the proof of Theorem 3.3.

Proof of Theorem 3.3. First assume that S_1 and U_1 are not zero.Then we must show that the point $(\overline{s}, 1, \overline{u}) = \left(\frac{G_1}{S_1}, 1, \frac{G_1}{U_1}\right)$ is the singularity of the cubic. A calculation shows $F_s(\overline{s}, 1, \overline{u}) = \Delta\Omega_1$, $F_t(\overline{s}, 1, \overline{u}) = \Delta\Omega_2$, $F_u(\overline{s}, 1, \overline{u}) = \Delta\Omega_3$, where the Ω_i are rational functions of the coefficients a, b, c, d, e and f whose denominators do not vanish because they are multiples of S_1 and U_1. So since $\Delta = 0$, it follows that $(\overline{s}, 1, \overline{u})$ is the singular point.

In the case $S_1 = 0$ or $U_1 = 0$ we will use Lemma 3.4 to show that the homogeneous coordinates of the singularity are given by (3.4) and (3.5).

Consider the case $S_1 = 0$. By Lemma 3.4, $G_1 = 0$, and since we have assumed F is irreducible, $S_2 \neq 0$ and $U_2 \neq 0$.

Suppose c and d are not zero. Solving for b in equation (3.7), we get

$$b = \frac{ce^3 f - ae^2 f^2 + 2c^2 de^2 - 12acdef + 12a^2 df^2 - 8ac^2 d^2}{cS_2}. \tag{3.9}$$

Substituting (3.9) in (3.1), Δ can be factored $(2a(af - ce)^2 \Delta_1^2)/(c^2 S_2^3)$, where

$$\begin{aligned}\Delta_1(a, c, d, e, f) =&108a^2 df^4 - 9ae^2 f^4 + 8ce^3 f^3 - 144acdef^3 \\ &- 144ac^2 d^2 f^2 + 48c^2 de^2 f^2 + 96c^3 d^2 ef + 64c^4 d^3.\end{aligned}$$

We know that $\Delta = 0$, but we need to know that for F in this case, $\Delta_1 = 0$. Either $af - ce = 0$ or $\Delta_1 = 0$. If $af - ce = 0$, we find from (3.9) that $bc = ad$. But then using Lemma 3.2 the cubic is reducible. Hence $\Delta_1 = 0$.

To show that formulas (3.4) and (3.5) determine the singular point, we check that the derivatives of $F(s, t, u)$ vanish at the point $\left(\frac{G_2}{S_2}, 1, \frac{G_2}{U_2}\right)$. Because this is difficult to check directly, we replace $\frac{G_2}{U_2}$ by $\frac{G_3}{H}$, where

$$G_3(a, b, c, d, e, f) = -f(8cef - 9af^2 + 4c^2 d)$$

and

$$H(a, b, c, d, e, f) = 2d(4c^2 d + 5cef - 6af^2)$$

This substitution is allowable (after we check that $H \neq 0$) because

$$G_1 | (G_3 U_2 - G_2 H). \tag{3.10}$$

Evaluating the partial derivatives of F at $(\overline{s}, 1, \overline{u})$, with $\overline{s} = \frac{G_2}{S_2}$ and $\overline{u} = \frac{G_3}{H}$, we obtain $F_s(\overline{s}, 1, \overline{u}) = \Delta_1\Psi_1$, $F_t(\overline{s}, 1, \overline{u}) = \Delta_1\Psi_2$ and $F_u(\overline{s}, 1, \overline{u}) = \Delta_1\Psi_3$, where Ψ_1, Ψ_2 and Ψ_3 are rational functions whose denominators are products of H and S_2. Because $\Delta_1 = 0$, it remains only to show that $H \neq 0$. From (3.10) it follows that if $H = 0$, then G_3 is zero. Since c and d are different from zero, $8cef - 9af^2 + 4c^2d = 0$ and $4c^2d + 5cef - 6af^2 = 0$, from which it is again possible to deduce that the cubic is reducible, for a contradiction.

If $c = 0$ or $d = 0$ the formulas (3.4) and (3.5) can be checked directly.

The demonstration when $U_1 = 0$ follows the same steps as when $S_1 = 0$, but using that

$$G_1 | (G_3(a,b,c,d,e,f)S_2(b,a,d,c,e,f) - H(a,b,c,d,e,f)G_2(b,a,d,c,e,f)) \quad \blacksquare$$

For the proof of (1) of Lemma 3.4, we assume that $S_1 = 0$. If $c \neq 0$ and $S_2 \neq 0$, again express b as in (3.9). After substituting this expression in (3.6), G_1 can be factored $((af - ce)\Delta_1)/cS_2$. Since $\Delta_1 = 0$, necessarily $G_1 = 0$.

If $c \neq 0$ and $S_2 = 0$, we can conclude first that $f \neq 0$. For $S_2 = 0$ and $f = 0$ together imply $d = 0$, which makes F reducible with factor s. Hence $S_2 = 0$ can be solved for a:

$$a = \frac{2c(ef + 2cd)}{3f^2}. \tag{3.11}$$

Substituting for a in S_1, we obtain $c(ef + 2cd)(4cd - ef)^2 = 0$. Since $a \neq 0$, this means $4cd - ef = 0$. Substituting for a in G_1, we obtain

$$G_1 = \frac{c(4cd - ef)(-4cd^2 - 5def + 6bf^2)}{3f} \tag{3.12}$$

so $G_1 = 0$.

If $c = 0$, then

$$S_1 = f^2a(12ad - e^2) \tag{3.13}$$

$$G_1 = f^2a(9bf - 8de) \tag{3.14}$$

From (3.13) we obtain

$$d = \frac{e^2}{12a}. \tag{3.15}$$

Using (3.15) and $c = 0$, Δ can be factored $f^2(27abf - 2e^3)^2/27$. Substituting (3.15) in (3.14), we obtain $G_1 = f^2(27abf - 2e^3)/3$. Since $\Delta = 0$, G_1 also is zero.

The proof in the case $U_1 = 0$ is analogous.

For (2), we know that $G_1 = 0$ and $S_2 = 0$. We may assume that c and f are different from zero, for otherwise using $S_2 = 0$ we see that F is reducible and we are done. Furthermore $d \neq 0$. For if $d = 0$, then G_1 can be factored $bf^2(9af - 8ce)$ and S_2 reduces to $f(3af - 2ce)$, hence $af = 0$, which contradicts $a \neq 0$ and $f \neq 0$. So we assume a, b, c, d and f are all nonzero.

Solving $S_2 = 0$ for a, we again obtain (3.11), and when we substitute for a in G_1 we obtain (3.12). One of the factors in the numerator of (3.12) must be zero.

If $ef - 4cd = 0$, we can deduce from $S_2 = 0$ that $af - ce = 0$. Substituting $a = ce/f$ into (3.1) and simplifying using $ef - 4cd = 0$, we obtain $-16c^3 f(de - bf)^3 = 0$. Now $af - ce = 0$ and $de - bf = 0$ imply that F is reducible by Lemma 3.2.

If the other factor of (3.12) is zero,

$$-4cd^2 - 5def + 6bf^2 = 0, \tag{3.16}$$

and

$$c = \frac{(6bf - 5de)f}{4d^2}. \tag{3.17}$$

Substituting (3.11) and (3.17) into the expression (3.1) for Δ, we obtain $\Delta = (27(6bf - 5de)f^2(2bf - de)^2(bf - de)^3)/4d^6$. We know $\Delta = 0$. If the first factor $6bf - 5de = 0$, then by (3.17) $c = 0$, a contradiction.

Combining the expression (3.8) for S_2 and (3.16), we obtain

$$cde + adf - 2bcf = 0. \tag{3.18}$$

If the second factor $2bf - de = 0$, then from (3.18) we obtain $adf = 0$, which is a contradiction.

Consequently, it can only happen that the third factor $bf - de = 0$. Using this and (3.18) we obtain also $ad - bc = 0$, so by Lemma 3.2, the cubic F is reducible. ■

§4 An Application

In this section we give the expressions for the control points and weights of the Bézier form of a singular cubic in terms of its double point $(\overline{s}, \overline{t}, \overline{u})$ and three of its coefficients a, b and f. Using these formulas together with Theorem 3.3, one can obtain the control points and weights in closed form entirely in terms of the coefficients.

Because we are working in a triangle, the control points can be assigned barycentric coordinates $b_0(1, 0, 0)$, $b_1(X, 1-X, 0)$, $b_2(0, 1-Y, Y)$ and $b_3(0, 0, 1)$. We refer to X and Y as the control points ratios of b_1 and b_2 respectively.

Theorem 4.1. *The control point ratios and weights of the Bézier cubic are*

$$X = \frac{a\overline{s}^2\overline{u} + f\overline{t}^3}{a\overline{s}^2\overline{u} + a\overline{s}\overline{t}\overline{u} + f\overline{t}^3}$$

$$Y = \frac{b\overline{s}\overline{u}^2 + f\overline{t}^3}{b\overline{s}\overline{u}^2 + b\overline{s}\overline{t}\overline{u} + f\overline{t}^3}$$

$$w_0 = -a\overline{s}\overline{u}^2$$

$$w_1 = \frac{1}{3}(a\overline{s}^2\overline{u} + a\overline{s}\overline{t}\overline{u} + f\overline{t}^3)$$

$$w_2 = \frac{1}{3}(b\overline{s}\overline{u}^2 + b\overline{s}\overline{t}\overline{u} + f\overline{t}^3)$$

$$w_3 = -b\overline{s}^2\overline{u}.$$

Proof: With the singular point $(\overline{s}, \overline{t}, \overline{u})$, three of the coefficients of F, given by (2.1), are

$$c = \frac{-(b\overline{s}\overline{u}^2 + f\overline{t}^3)}{\overline{s}\overline{t}^2}$$

$$d = \frac{-(a\overline{s}^2\overline{u} + f\overline{t}^3)}{\overline{t}^2\overline{u}}$$

$$e = \frac{-(2a\overline{s}^2\overline{u} + 2b\overline{s}\overline{u}^2 + f\overline{t}^3)}{\overline{s}\overline{t}\overline{u}}.$$

After these substitutions are made in F, a parametrization of $F = 0$ is obtained by the standard method of intersecting the curve with the family of lines through the double point. This parametrization is then converted to Bézier form. ∎

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M. Paluszny and F. Tovar
Laboratorio de Computación Gráfica y Geometría Aplicada
Facultad de Ciencias
Universidad Central de Venezuela
Apartado 47809, Los Chaguaramos
Caracas 1041-A. Venezuela
francisco.tovar@ciens.ucv.ve
marco@jade.ciens.ucv.ve

R. Patterson
Dept. of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 North Blackford Street
Indianapolis, Indiana, 46202-3216
rpatters@math.iupui.edu