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MATHEMATICI CELEBERRIMI,

ARS CONJECTANDI,

OPUS POSTHUMUM.

Accedit

TRACTATUS DE SERIEBUS INFINITIS,

Et EPISTOLA Gallicè scripta

DE LUDO PILÆ RETICULARIS.

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↃ IↃCC XIII.

TRACTATUS

DE

SERIEBUS INFINITIS

Earumque

Summa Finita,

ET

Uſu in Quadraturis Spatiorum

& Rectificationibus Curvarum.

PRAEFATIO.

CUm non ita pridem in Serierum Infinitarum speculationem incidissem, prima, cujus summa post Geometricam Progressionem ab aliis jam tractatam mihi sese offerebat, erat series fractionum, quarum denominatores Geometrica, numeratores Arithmetica progressione crescunt: quod cum Fratri indicassem, non tantum mox idem adinvenit ille, sed & præterea novæ cujusdam fractionum seriei, cujus denominatores Trigonalium, ut vocantur, numerorum dupli erant, summam pervestigavit; quam vero & ipse, cum significasset, postridie detexi, propositis ei vicissim aliis nonnullis, quæ interea, ut clavus clavum trudere solet, occasione hac repereram. Quibus inventis certatim alter alterum sic exercuimus, ut paucorum dierum spatio non tantum serierum illarum, quas Celeb. Leibnitius in Actis Erud. Lipsf. Anno 1682. M. Febr. & 1683. M. Octob. recenset, nosque paulo antea mirati fuimus, summas dare possemus, sed & plura alia eaque non contemnenda ex gemino duntaxat fundamento invenerimus, quorum unum consistit in resolutione seriei in alias infinitas series, alterum in subductione seriei uno alterove termino mutilatæ à seipsa integra. Horum vero præcipua (cum eorum nihil apud hos quos legi hactenus, publicatum viderim) enucleanda

proponam, præmissis nonnullis, quæ passim apud alios quoque vulgatæ prostant, Propositionibus, ne illas aliunde petere opus esset. Cæterum quantæ sit necessitatis pariter & utilitatis hæc serierum contemplatio, ei sane ignotum esse non poterit, qui perspectum habuerit, ejusmodi series sacram quasi esse anchoram, ad quam in maxime arduis & desperatæ solutionis Problematibus, ubi omnes alias humani ingenii vires naufragium passæ, velut ultimi remedii loco confugiendum est.

Axiomata seu Postulata.

1. OMne quantum est divisibile in partes se minores.
2. Omni quantitate finita potest accipi major.
3. Si quantitas quæpiam multata parte sui aliqua subtrahitur à seipsa integra, relinquitur illa pars.

PROPOSITIONES:

I. *Quod data quavis quantitate minus est, illud est non-quantum seu nihil.*
Dem. Nam si quantum esset, dividi posset in partes se minores, per Axiom. 1. non igitur esset data quavis quantitate minus, contra hyp.

II. *Quod data quavis quantitate majus est, infinitum est.*
Nam si finitum esset, illo posset accipi quantitas major, per Ax. 2. non igitur quavis data quantitate foret majus, contra hyp.

III. *Omnis Progressio Geometrica continuari potest per terminos infinitos.*
Semper enim fieri potest: Ut primus terminus ad secundum, sic postremus ad sequentem, & sequens ad alium & alium sine fine in infinitum; quorum quidem terminorum nullus æquari potest vel nihilo vel infinito, cum secus ad illum præcedens eam rationem habere non posset, quam habet primus ad secundum, contr. defin. progr.

IV. *Si sit Progressio Geometrica quæcunque* A, B, C, D, E; *& alia Arithmetica totidem terminorum* A, B, F, G, H, *incipiens ab iisdem terminis* A *& B,*

& B, erunt reliquorum singuli in Geometrica singulis ordine sibi respondentibus in Arithmetica majores, tertius tertio, quartus quarto, ultimus ultimo, adeoque omnes omnibus.

Quia enim $A . B :: B . C :: C . D :: D . E$. erit per 25. 5. Eucl. tum $A + C > 2B$ ∞ (ex nat. Progr. Arith.) $A + F$; unde $C > F$: tum $A + D > B + C > B + F$ ∞ $A + G$; unde $D > G$: tum $A + E > B + D > B + G$ ∞ $A + H$; unde $E > H$. Quæ erant demonstr.

V. *In progressione Geometrica crescente* A, B, C, D, E *perveniri tandem potest ad terminum* E *quovis dato* Z *majorem.*

Incipiat ab iisdem terminis Progressio Arithm. A, B, F, G, H, continuata quousque ultimus H superet Z (id enim fieri posse claret,) tum vero continuetur Geometrica per terminos totidem, eritque per præced. postremus $E > H > Z$. Q. E. D.

Coroll. Hinc in Progr. Geom. crescente infinitorum terminorum postremus terminus est ∞, per Prop. II. (∞ est Nota Infiniti.)

VI. *In Progress. Geometr. decrescente* A, B, C, D, E *pervenitur tandem ad terminum* E *quovis dato* Z *minorem.*

Constituatur Progressio ascendens Z, Y, X, V, T, in ratione B ad A, quousque ultimus terminus T superet A, (quod fieri posse per præced. constat;) tum continuetur altera descendendo per totidem terminos A, B, C, D, E; eritque ultimus $E <$ dato Z. Quia enim Progressiones A, B, C, D, E, & T, V, X, Y, Z, per eandem rationem A ad B progrediuntur, & terminos numero æquales habent, erit ex æquo $A . E :: T . Z$. sed $A < T$, per constr. Ergo & $E < Z$. Q. E. D.

Coroll. Hinc in Progr. Geomet. decrescente in infinitum continuata ultimus terminus est o, per Prop. I.

VII. *In omni Progr. Geom.* A, B, C, D, E, *primus terminus est ad secundum, sicut summa omnium excepto ultimo ad summam omnium excepto primo.* ($A . B :: A + B + C + D . B + C + D + E$.)

Quia enim $A . B :: B . C :: C . D :: D . E$. erit per 12. 5. Eucl. $A . B :: A + B + C + D . B + C + D + E$. Q. E. D.

VIII. *Progressionis Geom. cujuscunque* A, B, C, D, E, *summam* S *invenire.*

Per præc. est $A . B :: S - E . S - A$; quare convertendo $A . A = B :: S - E . A = E$; unde $S - E$ ∞ $\frac{A \text{ in } A = E}{A - B}$, & S ∞ $\frac{A \text{ in } A = E}{A - B} + E$. (= denotat differentiam duarum quantitatum, quibus interseritur, cum non definitur, penes utram sit excessus.)

Coroll. Si Progressio Geometr. descendendo continuetur in infinitum, adeoque ultimus terminus per Coroll. VI. evanescat, erit summa omnium $\frac{Aq}{A - B}$: unde liquet, quo pacto infiniti etiam termini finitam summam constituere possunt.

IX. *Si Series infinita continuè proportionalium* A, B, C, D, E, *&c. decrescat in ratione* A *ad* B, *erunt summæ omnium terminorum, omnium demto primo, omnium demtis duobus primis, &c. etiam continuè proportionales, & quidem in eadem ratione* A *ad* B.

Quoniam $A . B :: B . C :: C . D$, erit tum $Aq . Bq :: Bq . Cq$: tum etiam $A . B :: A - B . B - C :: B - C . C - D$, quare dividendo rationes æquales per æquales, $\frac{Aq}{A - B} . \frac{Bq}{B - C} :: \frac{Bq}{B - C} . \frac{Cq}{C - D}$, hoc est per Cor. præced. Summa omnium ad omnes sequentes primum, ut hi ad omnes sequentes secundum. Q. E. D. Et proinde per 19. 5. Eucl. summa omnium ad omnes sequentes primum, ut primus ad secundum. Q. E. D.

X. *Seriei infinitæ fractionum*, $\frac{a}{b}, \frac{a + c}{b + d}, \frac{a + 2c}{b + 2d}, \frac{a + 3c}{b + 3d}$, *&c. quarum numeratores & denominatores crescunt Progressione Arithmet. ultimus terminus est fractio* $\frac{c}{d}$, *cujus numerator & denominator sunt communes progressionum differentiæ.*

Ad hoc analyticè investigandum consideretur quæsitus terminus ut cognitus, & vocetur t; numerus vero termini ut quæsitus, & dicatur n; eritque ex generatione progressionis terminus optatus t ∞ $\frac{a + nc - c}{b + nd - d}$, ideoque n ∞ $1 + \frac{bt - a}{c - dt}$, quod æquari debet infinito: & quia numerator hujus fractionis est finitus (nam infinitus esse non potest, aliàs t deberet esse ∞ ∞; ideoque esset c = d, ipsaque adeo fractio negativa quantitas, quod absurdum,) oportet ut denominator sit æqualis nihilo, ac proinde c ∞ dt, & t ∞ $\frac{c}{d}$. Q. E. D.

Brevius ita: Ex seriei genesi patet, terminum infinitesimum esse $\frac{a + \infty c}{b + \infty d}$ ∞ $\frac{\infty c}{\infty d}$ ∞ $\frac{c}{d}$. Q. E. D.

Coroll. Summa omnium terminorum, sive ultimus primo major sit

sit minorve, necessario infinita est; infiniti enim termini minori horum duorum æquales infinitam dant summam: Unde à fortiori, &c.

XI. *Fractionis ad aliam ratio composita est ex ratione directa numeratorum & reciproca denominatorum.*

Nam $\frac{A}{B} \cdot \frac{C}{D} :: \frac{AD}{BD} \cdot \frac{BC}{BD} :: AD \cdot BC :: A \cdot C + D \cdot B$. Q. E. D.

XII. *In serie fractionam, quarum numeratores crescunt Progressione Arith. denominatores Geometrica, aut vice versa, ut* $\frac{A}{F} \cdot \frac{A+C}{G} \cdot \frac{A+2C}{H} \cdot \frac{A+3C}{I}$, aut $\frac{F}{A} \cdot \frac{G}{A+C} \cdot \frac{H}{A+2C} \cdot \frac{I}{A+3C}$: *Si* N *nomen ordinis ultimi termini ad unitatem majorem rationem habeat, quam* G *ad* G − F, *erit ille terminus ibi sequenti major, hîc minor.*

1. *Hyp.* Quia $N . 1 . > G . G-F$, erit convertendo $N . N-1 < G . F$, & $CN . CN-C < G . F$. Ergo $CN-C : in\ G > CN\ in\ F$, ergo fortius (ob $AG > AF$) $A+NC-C : in\ G > A+CN : in\ F$, hoc est, Numerator termini N in $G >$ Numeratore termini sequentis in F: Sed ita se habet terminus N ad terminum sequentem, per præced. Quare terminus N major sequenti, & ita deinceps ab illo omnes. Q. E. D.

2. *Hyp.* Inversis invertendis eodem modo demonstratur.

XIII. *Si infinitæ sint fractiones* $\frac{A}{B} \cdot \frac{C}{D} \cdot \frac{E}{F} \cdot \frac{G}{H} \cdot \frac{I}{L} \cdot \frac{M}{N} \cdot \frac{O}{P}$ *&c. quarum numeratores crescant progr. Arithm. & denominatores Geom. erit ultimus terminus* 0; *sin illi crescant Geometr. hi Arithm. erit ultimus* ∞.

1. *Hyp.* Si primus terminus secundo non sit major, continuari saltem poterit Progressio, quousque præcedens superet sequentem, per præced. Esto $\frac{G}{H} > \frac{I}{L}$, & sint infiniti continuè proportionales G, I, Q, R, &c. unde propter H, L, N, P ∺ erunt & ipsæ fractiones $\frac{G}{H} \cdot \frac{I}{L} \cdot \frac{Q}{N} \cdot \frac{R}{P}$ &c. ∺ quæ ob $\frac{G}{H} > \frac{I}{L}$. in nihilum tandem abeunt per Cor. VI. Quare cum $Q > M$, $R > O$, &c. per IV. multo magis $\frac{G}{H} \cdot \frac{I}{L} \cdot \frac{M}{N} \cdot \frac{O}{P}$, &c. in nihilum abibunt. Q. E. D.

2. *Hyp.* Nisi primus secundo minor sit, continuetur progressio, quousque præcedens sequenti minor fiat, per præced. Esto $\frac{G}{H} < \frac{I}{L}$, & sint

& sint infiniti H, L, S, T, &c. ∺ unde propter G, I, M, O, &c. ∺, & ipsæ fractiones $\frac{G}{H}, \frac{I}{L}, \frac{M}{S}, \frac{O}{T}$, &c. proportionales erunt, quæ ob $\frac{G}{H} < \frac{I}{L}$ in infinitum desinunt per Cor. V. Quare cùm $S > N$, $T > P$, &c. per IV. multo magis $\frac{G}{H}, \frac{I}{L}, \frac{M}{N}, \frac{O}{P}$, &c. in infinitum excrescent. Q. E. D.

XIV. *Invenire summam seriei infinitæ fractionum, quarum denominatores crescunt progressione Geometrica quacunque, numeratores verò progrediuntur vel juxta numeros naturales* 1, 2, 3, 4, *&c. vel trigonales* 1, 3, 6, 10, *&c. vel pyramidales* 1, 4, 10, 20, *&c. aut juxta quadratos* 1, 4, 9, 16, *&c. aut cubos* 1, 8, 27, 64, *&c. eorumve æquemultiplices.*

1. *Si Numeratores progrediuntur juxta numeros naturales:*

Summa invenitur, resolvendo seriem propositam A in alias infinitas series B, C, D, E, &c. quæ singulæ geometricè progrediuntur, quarumque summæ (si primam hîc excipias) novam Geometricam progressionem F constituunt per IX. cujus quidem, uti cæterarum, summa per Coroll. VIII. reperitur. En operationem:

A ꝏ $\frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bdd} + \frac{a+3c}{bd^3}$ &c. ꝏ B + C + D + E + &c.

B ꝏ $\frac{a}{b} + \frac{a}{bd} + \frac{a}{bdd} + \frac{a}{bd^3}$	&c. ꝏ $\frac{ad}{bd-b}$	
C ꝏ . $+ \frac{c}{bd} + \frac{c}{bdd} + \frac{c}{bd^3}$	&c. ꝏ $\frac{c}{bd-b}$.	F ꝏ $\frac{cd}{b\ in\ Q : d-1}$
D ꝏ . . . . $+ \frac{c}{bdd} + \frac{c}{bd^3}$	&c. ꝏ $\frac{c}{bdd-bd}$	
E ꝏ . . . . $+ \frac{c}{ba^3}$	&c. ꝏ $\frac{c}{bd^3-bdd}$	
&c. ꝏ	&c. ꝏ &c.	

cui additus primus terminus $\frac{ad}{bd-b}$ producit totius propositæ seriei A

$\frac{ad}{b\ in\ d-1} + \frac{cd}{b\ in\ Q : d-1}$ ꝏ summam.

2. *Si Numeratores sunt juxta Trigonales:*

Series proposita G resolvenda est in aliam H, cujus numeratores sint juxta præcedentem hypothesin, hoc modo:

G ꝏ

$$G \infty \frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bdd} + \frac{10c}{bd3} \&c.$$

$$\left.\begin{array}{r} \frac{c}{b} + \frac{c}{bd} + \frac{c}{bdd} + \frac{c}{bd3} \&c. \infty \frac{cd}{bd-b} \\ + \frac{2c}{bd} + \frac{2c}{bdd} + \frac{2c}{bd3} \&c. \infty \frac{2c}{bd-b} \\ + \frac{3c}{bdd} + \frac{3c}{bd3} \&c. \infty \frac{3c}{bdd-bd} \\ + \frac{4c}{bd3} \&c. \infty \frac{4c}{bd3-bdd} \\ \&c. \infty \&c. \end{array}\right\} H \infty \frac{cd3}{b \text{ in } C:d-1}$$

quandoquidem hæc ſeries ad præced. $\frac{c}{bd} + \frac{2c}{bdd} + \frac{3c}{bd3}$ &c. $\infty \frac{cd}{b \text{ in } Q:d-1}$ ſe habeat ut dd ad $d-1$.

3. *Si Numeratores ſunt juxta Pyramidales:*

Series reſolvitur in aliam, cujus numeratores progrediuntur juxta Trigonales, quæque ad præcedentem ſeriem ſe habet, ut d ad $d-1$; unde ſumma ejus invenitur $\infty \frac{cd4}{b \text{ in } QQ:d-1}$. Generaliter, ſi propoſitæ ſeriei numeratores ſint juxta figuratos cujuslibet gradus, ejus ſumma ſe habebit ad ſummam ſimilis ſeriei gradus præcedentis, ut d ad $d-1$: unde reliquarum omnium ſummam invenire proclive admodum eſt.

4. *Si Numeratores ſunt juxta Quadratos:*

Series L reſolvitur in aliam M, cujus numeratores ſunt Arithmeticè progreſſionales, adeoque juxta primam hypotheſin:

$$L \infty \frac{c}{b} + \frac{4c}{bd} + \frac{9c}{bdd} + \frac{16c}{bd3} \&c.$$

$$\left.\begin{array}{r} \frac{c}{b} + \frac{c}{bd} + \frac{c}{bdd} + \frac{c}{bd3} \&c. \infty \frac{cd}{bd-b} \\ + \frac{3c}{bd} + \frac{3c}{bdd} + \frac{3c}{bd3} \&c. \infty \frac{3c}{bd-b} \\ + \frac{5c}{bdd} + \frac{5c}{bd3} \&c. \infty \frac{5c}{bdd-bd} \\ + \frac{7c}{bd3} \&c. \infty \frac{7c}{bd3-bdd} \\ \&c. \infty \&c. \end{array}\right\} M \infty \frac{cdd}{b \text{ in } Q:d-1} + \frac{2cd}{b \text{ in } C:d-1} \infty \frac{cd3 + cdd}{b \text{ in } C:d-1}.$$

5. *Si Numeratores ſunt juxta Cubos:*

Series reſolvitur in aliam, cujus numeratores ſunt Trigonalium ſextupli unitate aucti; unde ejus ſumma juxta ſecundam hypotheſin

invenitur $\frac{cdd}{b \text{ in } Q:d-1} + \frac{6cd3}{b \text{ in } QQ:d-1} \infty \frac{cd4 + 4cd3 + cdd}{b \text{ in } QQ:d-1}$. Exempli loco ſint ſeries ſequentes, Numeratorum

Naturalium	$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32}$ &c. ∞ 2
Trigonalium	$\frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32}$ &c. ∞ 4
Pyramidalium	$\frac{1}{2} + \frac{4}{4} + \frac{10}{8} + \frac{20}{16} + \frac{35}{32}$ &c. ∞ 8
Quadratorum	$\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32}$ &c. ∞ 6
Cuborum . .	$\frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \frac{64}{16} + \frac{125}{32}$ &c. ∞ 26

Coroll. Patet, in omnibus hujusmodi ſeriebus poſtremos terminos in nihilum deſinere, & evaneſcere debere (quod ipſum jam præced. Propoſ. de earum una ex abundanti oſtendimus;) cùm alias illarum ſummæ finitæ eſſe non poſſent.

XV. *Invenire ſummam ſeriei infinitæ fractionum* R, *quarum numeratores conſtituunt ſeriem æqualium, denominatores vero Trigonalium, eorumve æquemultiplicium.*

Si à ſerie harmonicè proportionalium N, eademmet multata primo termino P ſubtrahatur, exoritur nova ſeries Q, cujus denominatores Trigonalium dupli ſunt, cujusque adeo ſumma æqualis erit ipſi primo termino ſeriei Harmonicæ N, per Ax. 3.

Operatio talis: A ſerie $N \infty \frac{a}{c} + \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \frac{a}{5c}$ &c.

ſubtracta ſeries $P \infty \frac{a}{2c} + \frac{a}{3c} + \frac{a}{4c} + \frac{a}{5c} + \frac{a}{6c}$ &c. $\infty N - \frac{a}{c}$

relinquit ſeriem $Q \infty \frac{a}{2c} + \frac{a}{6c} + \frac{a}{12c} + \frac{a}{20c} + \frac{a}{30c}$ &c. $\infty \frac{a}{c}$

cujus duplum $R \infty \frac{a}{c} + \frac{a}{3c} + \frac{a}{6c} + \frac{a}{10c} + \frac{a}{15c}$ &c. $\infty \frac{2a}{c}$

ſeries ſcil. fractionum propoſita, quarum denominatores ſunt numeri Trigonales, eorumve æque-multiplices.

Obſervandum tamen, non ſine cautela hac utendum eſſe methodo: Nam ſi à ſequente ſerie S eadem demto primo termino T ſubtrahatur, prodibit ſeries Q, quæ antea; nec tamen inde ſequitur, ſummam ſeriei Q, æqualem eſſe primo termino ſeriei $S \infty \frac{2a}{c}$. Cujus rei ratio eſt, quòd, ſi à ſerie S ſubtrahitur ſeries terminorum

minorum totidem T, in qua ſinguli termini poſtremum præcedentes ſingulos primum conſequentes in altera deſtruunt, reſiduum, hoc eſt reſultans ſeries Q, evidenter debet adæquari primo termino ſeriei S minus ultimo ipſius T; adeoque ipſi primo ſeriei S abſolutè æqualis eſſe nequit, niſi tum cùm ultimus ipſius T in nihilum deſinit, uti quidem deſinere perſpicuum eſt in ſerie P vel N: at non evaneſcit pariter in ſerie T vel S, verum eſt $\infty \frac{a}{c}$, per X. Quin itaque potius ſumma ſeriei $Q \infty \frac{2a}{c} - \frac{a}{c} \infty \frac{a}{c}$, ut ſuprà.

$$S \infty \frac{2a}{c} + \frac{3a}{2c} + \frac{4a}{3c} + \frac{5a}{4c} + \frac{6a}{5c} \text{ \&c.}$$
$$T \infty \frac{3a}{2c} + \frac{4a}{3c} + \frac{5a}{4c} + \frac{6a}{5c} + \frac{7a}{6c} \text{ \&c.}$$
$$Q \infty \frac{a}{2c} + \frac{a}{6c} + \frac{a}{12c} + \frac{a}{20c} + \frac{a}{30c} \text{ \&c.} \infty \frac{2a}{c} - \frac{a}{c} \infty \frac{a}{c}.$$

XVI. *Summa ſeriei infinitæ harmonicè progreſſionalium*, $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ *&c. eſt infinita.*

Id primus deprehendit Frater: inventa namque per præced. ſumma ſeriei $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$, &c. viſurus porrò, quid emergeret ex iſta ſerie, $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30}$, &c. ſi reſolveretur methodo Prop. XIV. collegit propoſitionis veritatem ex abſurditate manifeſta, quæ ſequeretur, ſi ſumma ſeriei harmonicæ finita ſtatueretur. Animadvertit enim,

Seriem A, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, &c. ∞ (fractionibus ſingulis in alias, quarum numeratores ſunt 1, 2, 3, 4, &c. transmutatis)

ſeriei B, $\frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \frac{6}{42}$, &c. ∞ C + D + E + F, &c.

$$\left.\begin{array}{l} C.\ \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}, \text{\&c.} \infty \text{ per præc. } \frac{1}{1} \\ D..\ + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}, \text{\&c.} \infty C - \frac{1}{2} \infty \frac{1}{2} \\ E\ ...\ + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}, \text{\&c.} \infty D - \frac{1}{6} \infty \frac{1}{3} \\ F\ .....\ + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}, \text{\&c.} \infty E - \frac{1}{12} \infty \frac{1}{4} \\ \qquad\qquad \text{\&c.} \infty \qquad \text{\&c.} \end{array}\right\} \infty G;$$

unde ſequitur, ſeriem $G \infty A$, totum parti, ſi ſumma finita eſſet.

Ego

Ego poſtmodum, cum indicaſſet, idem oſtenſivè hunc in modum: Summa ſeriei infinitæ harmonicæ $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, &c. ſuperat datum quemvis numerum. Ergo infinita eſt, per II. Eſto datus numerus N quantumcunque magnus: Abſcinde à principio ſeriei aliquot terminos, quorum ſumma æquet vel ſuperet unam unitatem numeri N, & à ſerie reliqua iterum aliquos abſcinde, quorum ſumma aliam unitatem numeri N ſuperet, idque ſi fieri poſſit repete toties, quot in numero N ſunt unitates; ſic termini abſciſſi omnes ſuperabunt totum numerum, multo magis igitur tota ſeries eundem ſuperabit. Si neges, abſciſſis aliquot reliquos unitatem ſuperare poſſe, eſto primus reliquorum, qui poſt abſciſſionem ultimam remanſerunt, $\frac{1}{a}$, & ſequentes $\frac{1}{a+1}, \frac{1}{a+2}, \frac{1}{a+3}$, &c. Conſtituatur ad duos primos terminos $\frac{1}{a}$ & $\frac{1}{a+1}$ Progreſſio Geometrica, cujus ideo ſinguli poſt ſecundum termini ſingulis reſpondentibus in Progreſſione Harmonica minores ſunt ob denominatores majores, per IV. & continuetur hæc uſque ad $\frac{1}{aa}$ (quod quidem fiet in terminis numero finitis propter a numerum finitum) eritque hæc ſeries Geometrica finita ∞ 1, per VIII. Harmonica itaque terminorum totidem ſuperabit unitatem. Q. E. D.

Coroll. 1. In propoſita ſerie initio ſumto à quolibet termino, erunt ab illo deinceps omnes, uſque ad illum, cujus locus deſignatur per quadratum numeri ordinis primi termini, ſimul ſumti unitate majores: ſic termini à 2do ad 4tum uſque unitatem ſuperant, hinc à 5to ad 25tum, hinc à 26 ad 676 (Q: 26) hinc à 677 ad 458329 (Q: 677) &c. Nam in Geometrica progreſſione termini his limitibus intercepti unitatem æquant; ergo in Harmonica ſuperant, ubi & plures intercipiuntur & majores; majores quidem uti vidimus; plures, quia denominatores terminorum, cùm ſint minores quàm in Geometrica per IV. tardiùs illos limites aſſequuntur.

2. Patet, omnem aliam ſeriem harmonicam infinitam, ſummam quoque exhibere infinitam; ut ex. gr. ſi loco $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ &c. proponatur $\frac{1}{1000} + \frac{1}{2000} + \frac{1}{3000} + \frac{1}{4000}$ &c. ubi ſinguli termini ſingulorum

gulorum

gulorum ſibi reſpondentium in altera, adeoque & omnes omnium, ſunt ſubmillecupli: nam infiniti pars milleſima & ipſa infinita eſt.

3. Summa ſeriei infinitæ, cujus poſtremus terminus evaneſcit, quandoque finita eſt, quandoque infinita.

4. Sequitur etiam, ſi modo in Geometriam ſaltum facere permiſſum eſt, ſpatium Curva Hyperbolica & Aſymptotis comprehenſum infinitum eſſe: Secta intelligatur Aſymptotos linea à centro A in partes æquales infinitas in punctis B, C, D, E, &c. è quibus ad curvam educantur rectæ totidem alteri Aſymptotôn parallelæ BM, CN, DO, EP, &c. & compleantur parallelogramma AM, BN, CO, DP, &c. quæ ob baſium æqualitatem inter ſe erunt, ut altitudines, ſeu ut rectæ BM, CN, DO, EP, &c. hoc eſt, ut $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. ex natura Hyperbolæ; cum igitur ſumma $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ &c. infinita oſtenſa ſit, erit & ſumma Parallelogrammorum AM, BN, CO, DP, &c. infinita, multoque magis ſpatium Hyperbolicum, quod Parallelogrammis illis circumſcriptum eſt.

XVII. *Invenire ſummam ſerierum Leibnitzianarum*, D. H. I. *aliarumque quarum denominatores ſunt numeri Quadrati aut Trigonales, minuti aliis Quadratis vel Trigonalibus.*

Cel. Leibnitius occaſione mirabilis ſuæ Quadraturæ Circuli in principio Actorum Lipſ. publicatæ, mentionem injicit ſummæ quarundam ſerierum infinitarum, quarum denominatores conſtituunt ſeriem quadratorum unitate minutorum, diſſimulato quo eam repererat artificio. En breviter totum myſterium:

A ſerie . . . . $A \infty \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ &c. ſubtrahatur ipſamet demtis duobus primis terminis, $B \infty \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$ &c. $\infty\ A-\frac{1}{1}-\frac{1}{2}$

relinquitur $C \infty \frac{2}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}+\frac{2}{35}$ &c. $\infty\ A-B \infty \frac{1}{1}+\frac{1}{2} \infty \frac{3}{2}$

& propterea $D \infty \frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}$ &c. $\infty\ \frac{1}{2}C \infty \frac{3}{4}$

A ſerie . . . $E \infty \frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$ &c. ſubtrahatur eadem demto primo termino, ... $F \infty \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}$ &c. $\infty\ E-1$

relin-

relinquitur $G \infty \frac{2}{3}+\frac{2}{15}+\frac{2}{35}+\frac{2}{63}+\frac{2}{99}$ &c. $\infty\ E-F \infty 1$

& propterea . . $H \infty \frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}$ &c. $\infty\ \frac{1}{2}G \infty \frac{1}{2}$,

& proinde etiam $I \infty \frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}$ &c. $\infty\ D-H \infty \frac{3}{4}-\frac{1}{2} \infty \frac{1}{4}$

Quod ipſum quoque ſic oſtenditur:

A ſerie . . . $L \infty \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$ &c. ſubtrahatur eadem demto primo termino, . . . $M \infty \frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$ &c. $\infty\ L-\frac{1}{2}$

relinquitur $N \infty \frac{2}{8}+\frac{2}{24}+\frac{2}{48}+\frac{2}{80}+\frac{2}{120}$ &c. $\infty\ L-M \infty \frac{1}{2}$

& proinde . . . $I \infty \frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}$ &c. $\infty \frac{1}{2}N \infty \frac{1}{4}$, ut anteà.

Memorabile autem prorſus eſt, quòd ſumma ſeriei D, $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{63}$ &c. (cujus denominatores ſunt numeri quadrati 4, 9, 16, 25, 36, &c. unitate minuti) invenitur $\frac{3}{4}$, quin & excerptis per ſaltum alternis terminis, ſumma ſeriei H, $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}$ &c. $\infty \frac{1}{2}$; at ſi ex hâc iterum ſimplici ſaltu terminos loco pari poſitos excerpas, ut relinquatur $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}$ &c. ejus ſeriei infinitæ ſumma eſt vera magnitudo circuli nullo numero exprimibilis, ſumto vid. quadrato diametri $\infty \frac{1}{2}$.

Cæterum generaliter invenire poſſumus ſummam cujuslibet ſeriei, cujus numeratores conſtituunt ſeriem æqualium, & denominatores ſeriem quadratorum minutorum communi aliquo quadrato Q. aut etiam ſeriem Trigonalium minutorum communi aliquo numero Trigonali T: ſi obſervemus, ejusmodi ſeries naſci per ſubductionem ſeriei harmonicæ truncatæ ab initio tot terminis (quot indicat ibi duplum radicis quadratæ communis quadrati Q, hîc duplum unitate auctum radicis trigonalis numeri trigonalis T) à ſe-ipſa integra:

Ex.gr. ad inveniendam ſummam ſeriei D, $\frac{1}{7}+\frac{1}{16}+\frac{1}{27}+\frac{1}{40}+\frac{1}{55}$ &c.
cujus denominatores ſunt quadrati, 16, 25, 36, 49, 64, 81, &c.
minuti communi Quadrato Q. . . 9, 9, 9, 9, 9, 9.
(cujus Radix Q. 3, & duplum 6.) 7, 16, 27, 40, 55, 72, &c.

A ſerie

A ſerie . . . . $A \infty \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ &c. ſubtrahatur eadem multata ſex primis terminis . . $B \infty \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}$ &c.

relinquitur . . . $C \infty \frac{6}{7} + \frac{6}{16} + \frac{6}{27} + \frac{6}{40} + \frac{6}{55} + \frac{6}{72}$ &c. $\infty A - B \infty$
$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \infty 2\frac{9}{20}$

adeoque . . . . $D \infty \frac{1}{7} + \frac{1}{16} + \frac{1}{27} + \frac{1}{40} + \frac{1}{55} + \frac{1}{72}$ &c. $\infty \frac{1}{6} C \infty \frac{49}{120}$.

Rurſus pro invenienda ſumma ſeriei E, $\frac{1}{4} + \frac{1}{9} + \frac{1}{15} + \frac{1}{22} + \frac{1}{30} + \frac{1}{39}$ &c.
cujus denominatores ſunt Trigonales 10, 15, 21, 28, 36, 45, &c.
minuti communi Trigonali T . . . . 6, 6, 6, 6, 6, 6, &c.
(cujus Radix Trigon. 3. & duplum unitate auctum 7) 4, 9, 15, 22, 30, 39, &c.

A ſerie . . . $A \infty \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ &c. ſubtrahatur eadem truncata ſeptem primis terminis
$F \infty \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14}$ &c.

relinquitur $G \infty \frac{7}{8} + \frac{7}{18} + \frac{7}{30} + \frac{7}{44} + \frac{7}{60} + \frac{7}{78} + \frac{7}{98}$ &c. $\infty A - F \infty$
$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \infty \frac{363}{140}$

adeoque . . $E \infty \frac{1}{4} + \frac{1}{9} + \frac{1}{15} + \frac{1}{22} + \frac{1}{30} + \frac{1}{39} + \frac{1}{49}$ &c. $\infty \frac{2}{7} G \infty \frac{363}{490}$

Atque ita per hanc Propoſitionem inveniri poſſunt ſummæ ſerierum, cum denominatores ſunt vel numeri Trigonales minuti alio Trigonali, vel Quadrati minuti alio Quadrato; ut & per XV. quando ſunt puri Trigonales, ut in ſerie $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15}$ &c. at, quod notatu dignum, quando ſunt puri Quadrati, ut in ſerie $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$ &c. difficilior eſt, quàm quis expectaverit, ſummæ perveſtigatio, quam tamen finitam eſſe, ex altera, qua manifeſto minor eſt, colligimus: Si quis inveniat nobiſque communicet, quod induſtriam noſtram eluſit hactenus, magnas de nobis gratias feret.

Hoc ſaltem monere adhuc liceat, quod ſpatium Hyperboloide Cubicali (cujus natura exprimitur per æquationem $xxy \infty aab$, hoc eſt, in qua Quadrata abſciſſarum ex Aſymptotis ſunt in applicatarum ratione

ratione reciproca,) & Aſymptotis ſuis comprehenſum, eodem modo ex finita hujus ſeriei ſumma finitum eſſe demonſtrari poſſit, quo ſimile ſpatium in ipſa Hyperbola ex infinita ſeriei Harmonicæ ſumma infinitum oſtenſum eſt.

XVIII. *Invenire ſummam ſeriei infinitæ reciprocæ numerorum Trigonalium, Pyramidalium, Trianguli-Pyramidalium, Pyramidi-Pyramidalium, & figuratorum altioris cujuſvis gradus in infinitum: atque infinitarum ſummarum ſummam.*

1. Quemadmodum ſi à ſerie fractionum harmonicè progreſſionalium, hoc eſt, ſerie reciproca numerorum naturalium A, eadem multata primo termino ſubtrahatur, naſcitur ſeries fractionum, quarum numeratores ſunt *unitates*, denominatores *trigonalium dupli*; ut patet ex demonſtr. XV. Ita ſi à ſerie reciproca trigonalium B, eadem truncata primo termino ſubducatur, exoritur ſeries fractionum, quarum numeratores progrediuntur juxta numeros naturales 2. 3. 4. 5. &c. ſed quæ reducuntur ad fractiones, quarum omnium numeratores ſunt *binarii*, denominatores vero *pyramidalium tripli*; unde ipſa ſeries ad ſeriem reciprocam pyramidalium C, ut $\frac{2}{3}$ ad 1. Pariter ſi à ſerie hac reciproca pyramidalium, ipſamet mutilata primo termino ſubducatur, relinquitur ſeries fractionum, quarum numeratores progrediuntur juxta numeros trigonales 3. 6. 10. 15. &c. ſed quæ reduci poſſunt ad alias, quarum numeratores omnes ſunt *ternarii*, denominatores vero *trianguli-pyramidalium quadrupli*, unde ipſa ſeries ad ſeriem reciprocam trianguli-pyramidalium D, ut $\frac{3}{4}$ ad 1: Et ſic deinceps in infinitum. Quocirca cum ſingulæ hæ per ſubductionem genitæ ſeries, quarum numeratores ſunt unitatum, denominatores figuratorum multipli, per Ax. 3. æquipolleant unitati, ipſæ figuratorum ſeries reciprocæ ordine dabunt ſummas, ut ſequitur:

A. Natur. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ &c. $\infty \frac{1}{0} \infty 1\frac{1}{0}$, per XVI.
B. Trigon. $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21}$ &c. $\infty \frac{2}{1} \infty 1\frac{1}{1}$, per XV.
C. Pyramid. $\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \frac{1}{56}$ &c. $\infty \frac{3}{2} \infty 1\frac{1}{2}$.
D. Triang. Pyr. $\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \frac{1}{126}$ &c. $\infty \frac{4}{3} \infty 1\frac{1}{3}$.
E. Pyr. Pyr. $\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \frac{1}{126} + \frac{1}{252}$ &c. $\infty \frac{5}{4} \infty 1\frac{1}{4}$.

2. Sum-

2. Summæ hæ à ſecunda ſerie ordine collectæ ſunt $1\frac{1}{1}$, $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$, &c. unde ſumma ſummarum eſt $1\frac{1}{1} + 1\frac{1}{2} + 1\frac{1}{3} + 1\frac{1}{4} +$ &c. quæ infinita eſt: quin & demtis ſingularum ſerierum primis terminis ſeu unitatibus, ſumma fit $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ &c. quæ itidem infinita exiſtit, per XVI. at demtis inſuper ſecundis terminis $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ &c. ſumma evadit finita & æqualis $1 + \frac{1}{2} \infty \frac{3}{2}$ per Axiom. 3.

XIX. *Invenire ſummam ſeriei finitæ reciprocæ Trigonalium, Pyramidalium, Trianguli- Pyramidalium, Pyram. Pyramidalium, & figuratorum altioris cujusvis gradus in infinitum.*

Poſito in qualibet ſerie numero terminorum n, poſtremi termini in ſeriebus directis numerorum naturalium, trigon. pyramid. (per ea quæ demonſtrabuntur alibi) ſunt ordine hi, qui ſequuntur: (denotantibus hîc & ubique punctulis continuam multiplicationem quantitatum, quibus interſeruntur.)

$$n,\ \frac{n \cdot n+1}{1 \cdot 2},\ \frac{n \cdot n+1 \cdot n+2}{1 \cdot 2 \cdot 3},\ \frac{n \cdot n+1 \cdot n+2 \cdot n+3}{1 \cdot 2 \cdot 3 \cdot 4},\ \text{\&c.}$$

& qui hos immediatè excipiunt, ſunt iſti:

$$n+1,\ \frac{n+1 \cdot n+2}{1 \cdot 2},\ \frac{n+1 \cdot n+2 \cdot n+3}{1 \cdot 2 \cdot 3},\ \frac{n+1 \cdot n+2 \cdot n+3 \cdot n+4}{1 \cdot 2 \cdot 3 \cdot 4}\ \text{\&c.}$$

ac propterea erunt ultimi termini in eorundem ſeriebus reciprocis iſti:

$$\frac{1}{n},\ \frac{1 \cdot 2}{n \cdot n+1},\ \frac{1 \cdot 2 \cdot 3}{n \cdot n+1 \cdot n+2},\ \frac{1 \cdot 2 \cdot 3 \cdot 4}{n \cdot n+1 \cdot n+2 \cdot n+3},\ \text{\&c.}$$

& qui hos immediatè ſequuntur,

$$\frac{1}{n+1},\ \frac{1 \cdot 2}{n+1 \cdot n+2},\ \frac{1 \cdot 2 \cdot 3}{n+1 \cdot n+2 \cdot n+3},\ \frac{1 \cdot 2 \cdot 3 \cdot 4}{n+1 \cdot n+2 \cdot n+3 \cdot n+4},\ \text{\&c.}$$

Jam ſi à qualibet ſerie reciproca eadem ipſa truncata ab initio & aucta in fine uno termino methodo Prop. XV. ſubtrahatur, ſubducto ſigillatim ſecundo termino à primo, tertio à ſecundo, ſequente ultimum ab ultimo, naſcitur ſeries terminorum totidem, quæ per ea quæ in præced. Propoſ. dicta ſunt, ſeriei reciprocæ figuratorum gradus ſequentis aut ſubdupla eſt, aut ſubſeſquialtera, aut ſubſeſquitertia, &c. atque inſuper per obſervata Propoſ. XV. æqualis primo termino minus ſequente ultimum ejus ſeriei, per cujus ſubductionem nata fuit: unde ipſa ſumma ſeriei finitæ reciprocæ figuratorum quorumcunque obtinetur facile, ut ſequitur:

B. Tri-

B. Trigon. $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}$ &c. uſque ad $\frac{1 \cdot 2}{n \cdot n+1} \infty \frac{2}{1} - \frac{2}{1}$ in $\frac{1}{n+1} \infty \frac{2}{1} - \frac{2}{n+1}$

C. Pyram. $\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$ &c. -- $\frac{1 \cdot 2 \cdot 3}{n \cdot n+1 \cdot n+2} \infty \frac{3}{2} - \frac{3}{2}$ in $\frac{1 \cdot 2}{n+1 \cdot n+2} \infty \frac{3}{2} - \frac{1 \cdot 3}{n+1 \cdot n+2}$

D. △. Pyr. $\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35}$ &c. $\frac{1 \cdot 2 \cdot 3 \cdot 4}{n \cdot n+1 \cdot n+2 \cdot n+3} \infty \frac{4}{3} - \frac{4}{3}$ in $\frac{1 \cdot 2 \cdot 3}{n+1 \cdot n+2 \cdot n+3} \infty \frac{4}{3} - \frac{1 \cdot 2 \cdot 4}{n+1 \cdot n+2 \cdot n+3}$

E. Py. Pyr. $\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56}$ &c. $\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{n \cdot n+1 \cdots n+4} \infty \frac{5}{4} - \frac{5}{4}$ in $\frac{1 \cdot 2 \cdot 3 \cdot 4}{n+1 \cdots n+4} \infty \frac{5}{4} - \frac{1 \cdot 2 \cdot 3 \cdot 5}{n+1 \cdots n+4}$

XX. *Invenire ſummam ſeriei infinitæ reciprocæ Trigonalium, Pyramidalium, Triang. Pyramidalium, &c. multatæ terminis initialibus quotlibet: & infinitarum ſummarum ſummam.*

1. Summa ſeriei infinitæ integræ Trigonalium, Pyramidalium, Triang. Pyramidalium, &c. eſt $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$, &c. per XVIII. ſi ex unaquaque ſerie ab initio abſcindantur n termini, ſumma abſciſſorum eſt $\frac{2}{1} - \frac{2}{n+1}$, $\frac{3}{2} - \frac{1 \cdot 3}{n+1 \cdot n+2}$, $\frac{4}{3} - \frac{1 \cdot 2 \cdot 4}{n+1 \cdot n+2 \cdot n+3}$, $\frac{5}{4} - \frac{1 \cdot 2 \cdot 3 \cdot 5}{n+1 \cdot n+2 \cdot n+3 \cdot n+4}$, &c. per XIX. ſubtracta ergo hac à ſumma omnium, erit ſumma reliquorum $\frac{2}{n+1}$, $\frac{1 \cdot 3}{n+1 \cdot n+2}$, $\frac{1 \cdot 2 \cdot 4}{n+1 \cdot n+2 \cdot n+3}$, $\frac{1 \cdot 2 \cdot 3 \cdot 5}{n+1 \cdot n+2 \cdot n+3 \cdot n+4}$, &c.

2. Summa ſerierum omnium mutilatarum ſeu nullo ſeu uno termino eſt infinita, duobus terminis eſt $\frac{3}{2}$ per XVIII. Hinc ſi demas tertios terminos (qui conſtituunt ſeriem trigonalium B truncatam duobus terminis, cujus ſumma per eandem eſt $\frac{2}{3}$) erit reliquorum omnium ſumma $\frac{3}{2} - \frac{2}{3} \infty \frac{5}{6} \infty \frac{5}{2 \cdot 3}$. Hinc denuo ſi quartos terminos auferas (qui formant ſeriem pyramidalium C itidem truncatam duobus terminis, ſummamque proin per præced. efficiunt $\frac{2}{8}$) relin-

relinquetur cæterorum omnium ſumma $\frac{5}{6} - \frac{2}{8} \infty \frac{7}{12} \infty \frac{7}{3.4}$. Hinc iterum ſi quintos terminos reſeces, exibit cæterorum ſumma $\frac{9}{4.5}$; ſi ſextos, $\frac{11}{5.6}$; ſeptimos, $\frac{13}{6.7}$; &c. adeoque univerſaliter ſi ex unaquaque ſerie tollantur n termini, erit mutilatarum ita ſerierum omnium ſumma reliqua $\frac{2n-1}{n-1.n}$.

Coroll. Series $\frac{2}{n+1} + \frac{1.3}{n+1.n+2} + \frac{1.2.4}{n+1.n+2.n+3} + \frac{1.2.3.5}{n+1.n+2.n+3.n+4}$ + &c. ſive, $\frac{2}{1}$ in $\frac{1}{n+1} + \frac{3}{2}$ in $\frac{1.2}{n+1.n+2} + \frac{4}{3}$ in $\frac{1.2.3}{n+1.n+2.n+3} + \frac{5}{4}$ in $\frac{1.2.3.4}{n+1.\text{——}.n+4}$ + &c. $\infty \frac{2n-1}{n-1.n}$: ſingula enim ſeriei hujus membra ſingulas figuratarum ſerierum mutilatarum ſummas exprimunt, per 1. part. hujus; adeoque & omnia omnium.

XXI. *Seriei hujus*, $\frac{1a}{1.2} + \frac{2a}{1.2.3} + \frac{3a}{1.2.3.4} + \frac{4a}{1.2.3.4.5}$ + &c: *hoc eſt*, $\frac{a}{2} + \frac{a}{1.3} + \frac{a}{1.2.4} + \frac{a}{1.2.3.5}$ + &c: *ſummam invenire.*

Series hæc naſcitur ſubductione ſequentis ſeriei, $\frac{a}{1} + \frac{a}{1.2} + \frac{a}{1.2.3} + \frac{a}{1.2.3.4}$ + &c: multatæ primo termino à ſeipſa integra, methodo Prop. XV. quare ejus ſumma ∞ a, primo ſc. termino hujus, per Axioma 3.

Coroll. Hinc $\frac{1}{1.2} + \frac{4}{1.2.3} + \frac{9}{1.2.3.4} + \frac{16}{1.2.3.4.5}$ + &c. (∞ F + G + H + I + &c.) $\infty \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4}$ + &c.

Nam F. $\frac{1}{1.2} + \frac{2}{1.2.3} + \frac{3}{1.2.3.4} + \frac{4}{1.2.3.4.5}$ + &c. $\infty \frac{1}{1}$ per XXI.

G. - - $+ \frac{2}{1.2.3} + \frac{3}{1.2.3.4} + \frac{4}{1.2.3.4.5}$ + &c. ∞ F $- \frac{1}{1.2} \infty \frac{1}{1.2}$

H. - - - - - $+ \frac{3}{1.2.3.4} + \frac{4}{1.2.3.4.5}$ + &c. ∞ G $- \frac{2}{1.2.3} \infty \frac{1}{1.2.3}$

I. - - - - - - - - - $+ \frac{4}{1.2.3.4.5}$ + &c. ∞ H $- \frac{3}{1.2.3.4} \infty \frac{1}{1.2.3.4}$

XXII.

XXII. *Invenire ſummas ſerierum K, L, M, N, quarum numeratores ſunt Arithmeticè progreſſionales, denominatores Trigonalium integrorum aut Quadratorum unitate minutorum quadrata.*

K $\infty \frac{3}{\square 1} + \frac{5}{\square 3} + \frac{7}{\square 6} + \frac{9}{\square 10} + \frac{11}{\square 15} + \frac{13}{\square 21}$ + &c.

L $\infty \frac{2}{\square 3} + \frac{3}{\square 8} + \frac{4}{\square 15} + \frac{5}{\square 24} + \frac{6}{\square 35} + \frac{7}{\square 48}$ + &c.

M $\infty \frac{1}{\square 3} + \frac{2}{\square 15} + \frac{3}{\square 35} + \frac{4}{\square 63} + \frac{5}{\square 99} + \frac{6}{\square 143}$ + &c.

N $\infty \frac{3}{\square 8} + \frac{5}{\square 24} + \frac{7}{\square 48} + \frac{9}{\square 80} + \frac{11}{\square 120} + \frac{13}{\square 168}$ + &c.

Per ſubductionem ſeriei $\frac{1}{\square 1} + \frac{1}{\square 2} + \frac{1}{\square 3} + \frac{1}{\square 4} + \frac{1}{\square 5} + \frac{1}{\square 6}$ + &c. mutilatæ primo termino à ſeipſa integra naſcitur ſeries aliqua, cujus termini ſunt ſubquadrupli terminorum reſpondentium ſeriei K; unde per Ax. 3. ſeries K ∞ 4 in $\frac{1}{\square 1} \infty$ 4.

Per ſubductionem vero ejuſdem ſeriei mutilatæ duobus primis terminis à ſeipſa integra oritur ſeries, quæ quadrupla eſt ſeriei L; unde per id. Ax. ſeries L $\infty \frac{1}{4}$ *in*: $\frac{1}{\square 1} + \frac{1}{\square 2} \infty \frac{5}{16}$.

Denique per ſubductionem ſeriei $\frac{1}{\square 1} + \frac{1}{\square 3} + \frac{1}{\square 5} + \frac{1}{\square 7} + \frac{1}{\square 9}$ + &c. multatæ primo termino à ſeipſa integra emergit alia, quæ octupla eſt ſeriei M, quare per 3. Ax. ſeries M $\infty \frac{1}{8}$ *in* $\frac{1}{\square 1} \infty \frac{1}{8}$: & propterea duplum ſeriei M, hoc eſt, omnes termini locorum imparium ſeriei L $\infty \frac{1}{4}$; adeoque reliqui termini ejuſdem ſeriei, hoc eſt, ipſa ſeries N $\infty \frac{5}{16} - \frac{1}{4} \infty \frac{1}{16}$.

XXIII. *Invenire ſummas ſerierum Q & R, item V & X, &c. quarum denominatores ſunt termini integri progreſſionis quadruplæ, noncuplæ, &c. numeratores vero termini progreſſionis duplæ, triplæ, &c. unitate tum minuti, tum aucti.*

Operatio talis:

O $\infty \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ + &c. ∞ 2 } per Cor. VIII.

P $\infty \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$ + &c. $\infty \frac{4}{3}$ }

Q ∞

$$Q \infty \frac{1-1\infty 0}{1} + \frac{2-1\infty 1}{4} + \frac{4-1\infty 3}{16} + \frac{8-1\infty 7}{64} + \frac{16-1\infty 15}{256} + \&c. \infty O - P \infty 2 - \tfrac{4}{3} \infty \tfrac{2}{3}$$

$$R \infty \frac{1+1\infty 2}{1} + \frac{2+1\infty 3}{4} + \frac{4+1\infty 5}{16} + \frac{8+1\infty 9}{64} + \frac{16+1\infty 17}{256} + \&c. \infty O + P \infty 2 + \tfrac{4}{3} \infty \tfrac{10}{3}$$

$$\left.\begin{array}{l} S \infty \frac{1}{1} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c. \infty \frac{3}{2} \\ T \infty \frac{1}{1} + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} + \frac{1}{6561} + \&c. \infty \frac{9}{8} \end{array}\right\} \text{per Corol. VIII.}$$

$$V \infty \frac{1-1\infty 0}{1} + \frac{3-1\infty 2}{9} + \frac{9-1\infty 8}{81} + \frac{27-1\infty 26}{729} + \frac{81-1\infty 80}{6561} + \&c. \infty S - T \infty \tfrac{3}{2} - \tfrac{9}{8} \infty \tfrac{3}{8}$$

$$X \infty \frac{1+1\infty 2}{1} + \frac{3+1\infty 4}{9} + \frac{9+1\infty 10}{81} + \frac{27+1\infty 28}{729} + \frac{81+1\infty 82}{6561} + \&c. \infty S + T \infty \tfrac{3}{2} + \tfrac{9}{8} \infty \tfrac{21}{8}$$

Idem inveniri poteſt, reſolvendo ſeries propoſitas Q, R; V & X methodo Prop. XIV. Exempli loco eſto ſeries

$$Q \infty \tfrac{1}{4} + \tfrac{3}{16} + \tfrac{7}{64} + \tfrac{15}{256} + \&c. \infty Y + Z + \Pi + \Sigma + \&c.$$

$$\left.\begin{array}{l} Y \infty \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \&c. \infty \text{ per Coroll. VIII. } \frac{1}{3} \\ Z \infty \text{- -} + \frac{2}{16} + \frac{2}{64} + \frac{2}{256} + \&c. \infty 2Y - \frac{2}{4} \infty \frac{2}{3} - \frac{2}{4} \infty \frac{1}{6} \\ \Pi \infty \text{- - -} + \frac{4}{64} + \frac{4}{256} + \&c. \infty 2Z - \frac{4}{16} \infty \frac{2}{6} - \frac{4}{16} \infty \frac{1}{12} \\ \Sigma \infty \text{- - - - -} + \frac{8}{256} + \&c. \infty 2\Pi - \frac{8}{64} \infty \frac{2}{12} - \frac{8}{64} \infty \frac{1}{24} \\ \&c. \infty \text{- - - - - - - - -} \&c. \text{- - - - - - - -} \infty \&c. \end{array}\right\} \tfrac{2}{3} \text{ per Coroll. VIII.}$$

XXIV. *In ſerie quavis infinita, cujus numeratores omnes ſunt æquales, denominatores vel numeri naturales, vel eorundem quadrata, cubi, aut alia quæcunque poteſtas, ſumma terminorum omnium in locis imparibus eſt ad ſummam omnium in paribus, ut ſimilis poteſtas binarii unitate mulctata ad unitatem.*

Putà in numeris naturalibus, ut 1 ad 1; in quadratis ut 3 ad 1; in cubis ut 7 ad 1; in biquadratis ut 15 ad 1; &c.

Modus inveſtigandi talis:

In Numeris Naturalibus:

Series iſta $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \&c.$ æquatur ſuis partibus, videl. ſeriebus $A + B + C + D + \&c.$

$$\left.\begin{array}{l} A \infty \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. \infty \frac{2}{1} \\ B \infty \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \&c. \infty \frac{2}{3} \\ C \infty \frac{1}{5} + \frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \&c. \infty \frac{2}{5} \\ D \infty \frac{1}{7} + \frac{1}{14} + \frac{1}{28} + \frac{1}{56} + \&c. \infty \frac{2}{7} \end{array}\right\} \text{per Cor. VIII.}$$

Eſt ergo $\frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \&c.$ æqualis, ideoque $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c.$ dimidia ſeriei propoſitæ $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ hoc eſt, ſumma terminorum in locis imparibus dimidia ſeriei totius, & proinde æqualis ſummæ reliquorum $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \&c.$

Patet hinc rurſum veritas Prop. XVI. cum enim $\frac{1}{1} > \frac{1}{2}$, $\frac{1}{3} > \frac{1}{4}$, $\frac{1}{5} > \frac{1}{6}$, &c. erit $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \&c. > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \&c.$ cui tamen æqualis modo oſtenſa eſt; quæ utique conciliari nequeunt, niſi ſumma utriusque ſeriei ſtatuatur infinita, hoc eſt, tanta ut quæ inter illas intercedit differentia, rationem æqualitatis deſtruere non poſſit.

In Numeris Quadratis:

Series $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \&c. \infty E + F + G + H + \&c.$

$$\left.\begin{array}{l} E \infty \frac{1}{1} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c. \infty \frac{4}{3.1} \\ F \infty \frac{1}{9} + \frac{1}{36} + \frac{1}{144} + \frac{1}{576} + \&c. \infty \frac{4}{3.9} \\ G \infty \frac{1}{25} + \frac{1}{100} + \frac{1}{400} + \frac{1}{1600} + \&c. \infty \frac{4}{3.25} \\ H \infty \frac{1}{49} + \frac{1}{196} + \frac{1}{784} + \frac{1}{3136} + \&c. \infty \frac{4}{3.49} \end{array}\right\} \text{per Cor. VIII.}$$

Eſt ergo $\frac{4}{3.1} + \frac{4}{3.9} + \frac{4}{3.25} + \frac{4}{3.49} + \&c. \infty \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \&c.$ adeoque prioris ſubſeſquitertia, hoc eſt, $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \&c.$ æqualis $\frac{3}{4}$ poſterioris, hoc eſt, termini omnes locorum imparium in ſerie propoſita conſtituunt tres quartas partes totius ſeriei, & reliqui unam: quare ſumma terminorum illorum ad ſummam horum, ut 3 ad 1.

Eadem inveſtigandi methodus obſervetur in reliquis poteſtatibus.

Aliter & univerſaliter ita:

$$x \infty \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} \&c.$$

$$y \infty \frac{1}{1^m} \qquad + \frac{1}{3^m} \qquad + \frac{1}{5^m} \qquad \&c.$$

$$x-y \infty +\frac{1}{2^m}\left(\frac{1}{2^m 1^m}\right)+\frac{1}{4^m}\left(\frac{1}{2^m 2^m}\right)+\frac{1}{6^m}\left(\frac{1}{2^m 3^m}\right) \&c.$$

$$2^m x-2^m y \infty +\frac{1}{1^m} \quad +\frac{1}{2^m} \quad +\frac{1}{3^m} \&c. \ \infty\ x$$

unde $2^m x - x \infty 2^m y$, & $y \infty x-\frac{x}{2^m}$, & $x-y \infty \frac{x}{2^m}$, ergo y. $x-y :: x-\frac{x}{2^m} . \frac{x}{2^m} :: 1-\frac{1}{2^m} . \frac{1}{2^m} :: 2^m - 1 . 1.$

Schol. Liquet hinc, quod ſummæ duarum ſerierum (etiamſi incógnitæ) poſſint ad ſe invicem habere rationem cognitam. vid. Prop. XVII. ſub fin. Extendit ſe autem demonſtratio ad poteſtatum radices ſive ad poteſtates fractas non minus ac integras: ſic ex. gr. colligimus, in ſerie $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{8}}+\frac{1}{\sqrt{27}}+\frac{1}{\sqrt{64}}+\frac{1}{\sqrt{125}}+$ &c. (ubi denominatores ſunt cuborum radices quadratæ) omnes terminos locorum imparium ad omnes parium eſſe, ut $\sqrt{8}-1$ ad 1. Mirabile verò eſt, quod in ſerie $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+$ &c. (cujus ſumma infinita eſt, ceu major ſerie $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. ob denominatores minores) termini locorum imparium ad terminos parium juxta regulam inveniuntur habere rationem $\sqrt{2}-1$ ad 1. minoris ſc. ad majus, cùm tamen illi cum his ſigillatim collati iisdem manifeſto ſint majores: cujus ἐναντιοφανείας rationem, etſi ex infiniti natura finito intellectui comprehendi non poſſe videatur, nos tamen ſatis perſpectam habemus. Idem vero de ſimilibus ſeriebus aliis, quæ infinitam ſummam habent, intelligendum.

XXV. *Series Theſis X*, $\frac{a}{b}-\frac{a+c}{b+d}+\frac{a+2c}{b+2d}-\frac{a+3c}{b+3d}$; *& alia Harmonica terminorum totidem & denominatorum eorundem*, $\frac{f}{b}-\frac{f}{b+d}+\frac{f}{b+2d}-\frac{f}{b+3d}$; *ſignis + & — alternatim ſe excipientibus, ſumtoque* $f \infty a-\frac{bc}{d}$, *æquales ſummas habent.*

Etenim

Etenim ſubtrahendo terminos locorum parium à terminis imparium, provenit eadem utrobique ſeries, $\frac{ad-bc}{bb+bd}+\frac{ad-bc}{bb+5bd+6dd}$, ſive $\frac{df}{bb+bd}+\frac{df}{bb+5bd+6dd}$, &c.

Eſto ex.gr.ſeries th.X. $\frac{3}{1}-\frac{5}{2}+\frac{7}{3}-\frac{9}{4}+\frac{11}{5}-\frac{13}{6}$, poſitoq; $f \infty 3-2 \infty 1$,

ſeries harmonica, $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$, erit

ſumma utriusque $\frac{1}{2} + \frac{1}{12} + \frac{1}{30}$, per ſaltum excerpta ex ſerie Q. th. XV.

XXVI. *Seriei infinitæ fractionum* K (*quarum denominatores creſcunt progreſſione Geometrica, hoc eſt, ſequentes præcedentium ſunt æque-multiplices exactè, numeratores verò præcedentium æque-multiplices aucti vel minuti communi quodam numero, ſummam ultimumve terminum reperire.*

($\pm$ denotat vel ubique + vel ubique —)

$$K \infty \frac{a}{c}+\frac{ab \pm d}{cm}+\frac{abb \pm bd \pm d}{cmm}+\frac{ab3 \pm bbd \pm bd \pm d}{cm3}+\frac{ab4 \pm b3d \pm bbd \pm bd \pm d}{cm4}+ \&c.$$

1. Summa ſeriei invenitur, reſolvendo illam methodo Prop. XIV. in ſeries fractionum purè proportionalium L+M+N+O+P+ &c.

$$\left.\begin{array}{l}
L \infty \frac{a}{c}+\frac{ab}{cm}+\frac{abb}{cmm}+\frac{ab3}{cm3}+\frac{ab4}{cm4}+ \&c. \infty + \frac{am}{m-b : \text{in } c} \\
M \infty \text{ - - } \pm \frac{d}{cm} \pm \frac{bd}{cmm} \pm \frac{bbd}{cm3} \pm \frac{b3d}{cm4} \pm \&c. \infty \pm \frac{d}{m-b : \text{in } c} \\
N \infty \text{ - - - } \pm \frac{d}{cmm} \pm \frac{bd}{cm3} \pm \frac{bbd}{cm4} \pm \&c. \infty \pm \frac{d}{m-b : \text{in } mc} \\
O \infty \text{ - - - - - - } \pm \frac{d}{cm3} \pm \frac{bd}{cm4} \pm \&c. \infty \pm \frac{d}{m-b : \text{in } mmc} \\
P \infty \text{ - - - - - - - - - } \pm \frac{d}{cm4} \pm \&c. \infty \pm \frac{d}{m-b : \text{in } m3c} \\
\&c. \infty \text{ - - - - - - - - - - - - } \pm \&c. \infty \pm \quad \&c.
\end{array}\right\} \text{per Cor. VIII.}$$

Summæ ſerierum M, N, O, P, &c. novam progreſſionem Geometricam conſtituunt, cujus ſumma per Coroll. VIII. eſt $\frac{md}{m-1 : \text{in } m-b : \text{in } c}$, quæ ſummæ ſeriei L $\frac{am}{m-b : \text{in } c}$ addita vel ſubtracta efficit $\frac{amm - am \pm md}{m-1 : m-b : \text{in } c}$ ſummam omnium ſerierum L, M, N, &c. hoc eſt, ipſius ſeriei propoſitæ K.

2. Obſer-

2. Obſervandum, ſi $m > b$, ſummam eſſe finitam, adeoque ultimum ſeriei terminum evaneſcere, vid. Cor. XIV.

Sin $m < b$, & ſumma infinita eſt, & ultimus quoque terminus eſt infinitus; tum enim ſingulæ progreſſiones Geometricæ L, M, N, &c. ſunt creſcentes: confer Prop. V.

At exiſtente $m \infty b$, ſumma quidem infinita eſt, ſed poſtremus terminus finitus: tum enim ſurrogato m in locum b, ſecundus terminus fit $\frac{am \pm d}{cm}$, hoc eſt, $\frac{a}{c} \pm \frac{d}{cm}$: tertius $\frac{amm \pm md \pm d}{cmm}$, hoc eſt, $\frac{a}{c} \pm \frac{d}{cm} \pm \frac{d}{cmm}$: quartus $\frac{am3 \pm mmd \pm md \pm d}{cm3}$, hoc eſt, $\frac{a}{c} \pm \frac{d}{cm} \pm \frac{d}{cmm} \pm \frac{d}{cm3}$: atque ita poſtremus $\frac{a}{c} \pm \frac{d}{cm} \pm \frac{d}{cmm} \pm \frac{d}{cm3} \pm \frac{d}{cm4} \pm$ &c. in infinitum: unde patet, terminum infiniteſimum reſolvi in $\frac{a}{c} \pm$ ſerie infinitorum Geometricè progreſſionalium in ratione m ad 1, quorum ſumma per Cor. VIII. eſt $\frac{d}{m-1 : \text{in } c}$, quæ ipſi $\frac{a}{c}$ addita vel ſubtracta efficit terminum infiniteſimum $\frac{am - a \pm d}{m-1 : \text{in } c}$, cujus numerator differentiam numeratorum primi & ſecundi termini, uti & denominator denominatorum eorundem differentiam exprimit: quare cùm ex Prop. X. manifeſtum ſit, terminum ultimum hujus progreſſionis

$$Q \infty \frac{a}{c}, \frac{am \pm d}{cm}, \frac{2am - a \pm 2d}{2cm - c}, \frac{3am - 2a \pm 3d}{3cm - 2c}, \frac{4am - 3a \pm 4d}{4cm - 3c}, \text{\&c.}$$

$$\text{ſive } \frac{a}{c}, \frac{a}{c} \pm \frac{d}{cm}, \frac{a}{c} \pm \frac{2d}{2cm - c}, \frac{a}{c} \pm \frac{3d}{3mc - 2c}, \frac{a}{c} \pm \frac{4d}{4cm - 3c}, \text{\&c.}$$

itidem eſſe $\frac{am - a \pm d}{m-1 : \text{in } c}$ ſive $\frac{a}{c} \pm \frac{d}{m-1 : \text{in } c}$; ſequitur in utraque progreſſione K & Q, primis duobus terminis exiſtentibus iisdem ultimos quoque eſſe pares, quamvis incrementa vel decrementa prioris magis ſubitanea ſint, quandoquidem ejus termini non niſi per ſaltum ex poſteriore ſunt excerpti: Invenio enim, quod memorabile eſt, tertium terminum ſeriei K convenire cum termino $m + 2$, quartum cum $mm + m + 2$, quintum cum $m3 + mm + m + 2$, ſextum cum $m4 + m3 + mm + m + 2$ ſeriei Q, & ſic deinceps; uti patere poterit ex ſubjunctis ſeriebus, ubi a valet 2, c 3, b vel m 3, & d 1.

$K \infty \frac{2}{3}, \frac{7}{9}, \frac{22}{27}, \frac{67}{81}, \frac{202}{243}$, &c. ultimus $\frac{5}{6}$.

$Q \infty \frac{2}{3}, \frac{7}{9}, \frac{12}{15}, \frac{17}{21}, \frac{22}{27}, \frac{27}{33}, \frac{32}{39}, \frac{37}{45}, \frac{42}{51}, \frac{47}{57}, \frac{52}{63}, \frac{57}{69}, \frac{62}{75}, \frac{67}{81}$, &c. ultimus $\frac{5}{6}$.

Intellige vero, quæ dicta ſunt de ſumma ultimoque termino ſeriei K, ſi numeratores præcedentium ſunt æque-multiplices aucti communi numero d; vel diminuti quidem eodem numero, at inſuper $ab > a + d$. Nam ſi ſit $ab \infty a + d$, æquivalebunt ſinguli numeratores ipſi a, ſummaque ſeriei fiet finita, nempe $\frac{am}{m-1 : \text{in } c}$, & ultimus terminus evaneſcet, ſive m exiſtat $<$ vel ∞ ipſi b.

XXVII. *Si dati cujuslibet numeri radix quadrata ducatur in ipſum numerum, & producti radix quadrata denuo ducatur in eundem, & producti hujus radix iterum iterumque; idque fiat continuo in infinitum: erit radix producti ultimi æqualis ipſi dato numero:* (putà, ſi datus numerus vocetur a, erit $\sqrt{a\sqrt{a\sqrt{a\sqrt{a\sqrt{a}}}}}$ &c. ∞a.)

Ponatur enim $x \infty \sqrt{: a\sqrt{: a\sqrt{: a\sqrt{: a}}}}$ &c. erit $xx \infty a\sqrt{: a\sqrt{: a\sqrt{: a}}}$ &c. & $\frac{xx}{a} \infty \sqrt{: a\sqrt{: \sqrt{: a}}}$ &c. ∞x: proinde $xx \infty ax$, & $x \infty a$. Q. E. D.

XXVIII. *Si dati numeri cujuslibet radix quadrata addatur ipſi dato numero, & aggregati radix quadrata denuo addatur eidem, & aggregati hujus radix iterum iterumque; idque fiat continuo in infinitum: radix aggregati ultimi radicem dati numeri quarta parte unitatis aucti dimidia unitate ſuperabit.* (putà $\sqrt{: a + \sqrt{: a + \sqrt{: a + \sqrt{: a +}}}}$ &c. $\infty \frac{1}{2} + \sqrt{\frac{1}{4} + a}$.)

Poſito enim $x \infty \sqrt{: a + \sqrt{: a + \sqrt{: a +}}}$ &c. erit $xx \infty a + \sqrt{: a + \sqrt{: a +}}$ &c. & $xx - a \infty \sqrt{: a + \sqrt{: a + \sqrt{: a +}}}$ &c. ∞x: proinde $xx \infty x + a$, & $x \infty \frac{1}{2} + \sqrt{\frac{1}{4} + a}$. Q. E. D.

XXIX. *Datis duobus numeris quibusvis, ſi radix quadrata unius ducatur in alterum, & producti radix quadrata in primum, & hujus producti radix in alterum; atque ita ſemper productorum radices ducantur alternatim in datorum alterum; idque continuetur in infinitum: erit radix producti ultimi æqualis alterutri duorum mediorum proportionalium inter duos datos numeros* (putà ſi dati numeri dicantur a & b, erit $\sqrt{: a\sqrt{: b\sqrt{: a\sqrt{: b\sqrt{: a\sqrt{: b}}}}}}$. &c. $\infty \sqrt{C.\, aab}$.)

Esto namque x ꝏ √: a √: b √: a √: b. &c. erit xx ꝏ a √: b √: a √: b &c. & $\frac{xx}{a}$ ꝏ √: b √: a √: b. &c. & $\frac{x^4}{aa}$ ꝏ b √: a √: b. &c. & $\frac{x^4}{aab}$ ꝏ √: a √: b. &c. ꝏ x: proinde x^4 ꝏ $aabx$, & x^3 ꝏ aab, & x ꝏ √C. aab. Q. E. D.

XXX. *Datis duobus numeris quibusvis, si radix cubica producti ex utroque ducatur in eorum primum, & producti radix quadrata ducatur in productum ex utroque, & hujus producti radix cubica denuo in eorum primum; & sic alternatim radices cubicæ & quadratæ ducantur in eorum primum & productum ex utroque, erit radix producti ultimi æqualis primo vel secundo quatuor mediorum proportionalium inter duos datos* (putà √: a √C: a b √: a √C: a b, &c. ꝏ √S. $a^4 b$, & √C: a b √: a √C: a b √: a &c. ꝏ √S. $a^3 bb$.)

XXXI. *Datis duobus numeris quibusvis, si radix quadrata secundi ducatur in primum, & producti radix quadrata iterum in primum, producti vero hujus radix in secundum, & hujus producti radix denuò in primum, & sic alternatim productorum radices multiplicentur, bis in primum, semel in secundum, erit radix producti ultimi* ꝏ *primo, secundo vel quarto sex proportionalium inter duos datos* (putà √: a √: a √: b √: a √: a √: b &c. ꝏ √BS. $a^6 b$, & √: a √: b √: a √: a √: b √: a √: a &c. ꝏ √BS. $a^5 bb$, & √: b √: a √: a √: b √: a √: a &c. ꝏ √BS. $a^3 b^4$.)

XXXII. *Datis duobus numeris quibusvis* p *&* q, *si tertius quicunque ductus in* q, *addatur ipsi* pp, *& ex radice summæ subtrahatur* p, *& residui radix in* q *ducta addatur ipsi* pp, *& ex radice summæ denuò subtrahatur* p, *& sic deinceps in infinitum, erit radix ultimi residui,* putà √: − p + √: pp + q √: − p + √: pp + q √: − p + &c. *radix æquationis cubicæ* x^3 ꝏ − 2 p x + q.

XXXIII. *Iisdem positis, quæ in præcedente, si subtractio ipsius* p *vertatur in additionem, erit radix aggregati ultimi,* putà √: + p + √: pp + q √: + p + √: pp + q √: &c. *radix æquationis* x^3 ꝏ + 2 p x + q.

XXXIV. *Datis duobus numeris* p *&* q, *si tertius in* q *ductus subtrahatur à* pp, *& radix reliqui ad* p *addatur dematurve, & summæ reliquique radix in* q *ducta subducatur à* pp, *& radix reliqui,* &c. *erunt radices summæ residuique ultimi,* putà √: p ± √: pp − q √: p ± √: pp − q √: &c. *radices æquationis* x^3 ꝏ + 2 p x − q.

XXXV. *Non secus datis tribus numeris* p, q, r, *erit* √: − p + √: pp + r + q √: − p + √: pp + r + q &c. *radix æquationis biquadraticæ* x^4 ꝏ − 2 p x x + q x + r.

Omnes hæ Propp. ad eundem modum demonstrantur, quo Propp. XXVII. XXVIII. & XXIX. quorsum itaque κοκκύζειν!

Schol. Patet hinc aditus ad inventionem 2. med. proport. & in genere radicum Problematum solidorum & hypersolidorum per solas rectas lineas & circulos, quam præstantissimi omnium seculorum Geometræ à bis mille retro annis anxiè sed frustrà quæsivere. Hanc ego, quoad fieri potuit, per seriem constructionis in infinitum continuandæ, primus omnium exhibui in Actis Lips. mens. Septemb. 1689. cum nemo simile quicquam scripto publicasset, fortè nec animo concepisset uspiam.

De Usu Serierum Infinitarum in Quadraturis Spatiorum & Rectificationibus Curvarum.

Postquam prima parte laboris nostri defuncti sumus, variarumque quoad fieri potuit, serierum summas exhibuimus, superest, ut ad alteram instituti partem transeamus, ostendendo modum eas applicandi ad dimensiones quantitatum Geometricarum, præsertim illarum, quas transcendentes *nuncupant, licet seriebus, quæ hic usui venient, raro contingat esse ex numero earum, quas proximè contemplati sumus, quarumque summas in potestate habemus. Observarunt enim Geometræ, plurimas dari quantitates, cujusmodi sunt pleræque Lineæ Curvæ, & pleraque ab iis comprehensa spatia, quæ nullis numeris vel rationalibus vel surdis quantumvis compositis exprimi, h. e. quarum relationes ad alias datas sub nulla æquatione algebraica definiti gradus cogi possent, sed quæ omnes æquationum gradus quasi transcenderent; ac idcirco attentandum duxerunt, num quas uno aliquo numero effari non poterant, per seriem saltem infinitorum, maximè rationalium, exprimere liceret, quibus ita continuò ad quæsitum accederetur, ut error tandem data quavis quantitate minor fieret, totaque series exactum quæsiti valorem exhiberet. Inventum, quod, quantum constat, vergente demum hoc seculo à* Mercatore, Gregorio, Newtono, Leibnitio, *in lucem productum fuit. Quid primi tres de his memoriæ prodiderint, etiamnum ignoramus. Summus Geometra Leibnitius, qui rem haud dubiè longissimè provexit, inter alias series, quas nobis in Actis Lipſ. impertivit, unam initio Actorum 1682. pro Circuli magnitudine dedit ſed methodum, qua illuc pervenit, nusquam expoſuit. Quantum conjicio, non differt illa à noſtra; nam & in easdem cum ipſo ſeries incidimus & ipſius ſubinde calculo differentiali uſi ſumus, uti poſthac patebit. Principia hujus calculi exponere nimis longum & alienum foret. Ea Vir perilluſtris D. Marchio* Hoſpitalius *in Libro de Analyſi infinitè parvorum nuperrimè edito perſpicuè tradit, ad quem proin Lectorem* φιλομαθῆ *remittimus.*

Definitio :

Mixtam Seriem voco, cujus termini multiplicatione ſunt conflati ex terminis ejusdem ordinis aliarum ſerierum. Ita ſi ſint ſeries *a*, *b*, *c*, *d*, *e*, &c. & *f*, *g*, *h*, *i*, *k*, &c. mixta ex utraque erit *af*, *bg*, *ch*, *di*, *ek*, &c.

XXXVI.

Fractionem $\frac{l}{m-n}$ *convertere in ſeriem infinitam quantitatum geometricè proportionalium.*

Fit hoc per diviſionem continuam numeratoris per denominatorem, hoc pacto: m in l habeo $\frac{l}{m}$, quod multiplicatum per diviſorem $m-n$, & ſubtractum ex dividendo l relinquit $\frac{ln}{m}$; hoc rurſus diviſum per m facit $\frac{ln}{mm}$, quod ductum in $m-n$ & ſubtractum ex dividendi reliquo efficit reſiduum $\frac{lnn}{mm}$; hoc denuò diviſum per m, facit $\frac{lnn}{m^3}$, quo ducto in $m-n$ & ſubtracto remanet $\frac{ln^3}{m^3}$, atque ita deinceps ſine fine in infinitum: ſemper enim aliquid dividendum ſupereſt, cum unius membri dividendus à diviſore bimembri nunquam ſine reſiduo exhauriri poſſit. At hoc reſiduum, continuata operatione poſitoque $m > n$, perpetuo decreſcit, & tandem data quavis quantitate minus fit, ut patet. Eſt ergo fractio propoſita

$\frac{l}{m-n}$

$\frac{l}{m-n} \infty \frac{l}{m} + \frac{ln}{mm} + \frac{lnn}{m^3} + \frac{ln^3}{m^4} + \frac{ln^4}{m^5}$ &c. quæ ſeries eſt quantitatum geometricè progredientium in ratione m ad n; quandoquidem quilibet ejus terminus ex conſtructione in n ductus & per m diviſus proximè ſequentem exhibet.

Idem brevius ſic evincitur: Summa Progreſſionis Geometricæ $\frac{l}{m} + \frac{ln}{mm} + \frac{lnn}{m^3} + \frac{ln^3}{m^4} + \frac{ln^4}{m^5}$ &c. eſt $\frac{l}{m-n}$, per Corollar. VIII. Ergo reciprocè valorem fractionis $\frac{l}{m-n}$ per talem ſeriem exprimere licet.

XXXVII. *Fractionem* $\frac{l}{m+n}$ *reſolvere in ſeriem infinitam geometricè proportionalium.*

Facta diviſione continua numeratoris per denominatorem, eadem reſultat ſeries, quæ anteà, niſi quod termini ejus alternatim fiant poſitivi & negativi. Eſt igitur quantitas $\frac{l}{m+n} \infty \frac{l}{m} - \frac{ln}{mm} + \frac{lnn}{m^3} - \frac{ln^3}{m^4} + \frac{ln^4}{m^5}$ &c. ſaltem ſi ponatur $m > n$: tum enim quod poſt ſingulas diviſiones reliquum manet, continuo minuitur, donec continuata in infinitum operatione prorſus evaneſcat.

Idem quoque ſic eluceſcit: Quoniam in ſerie quantitatum $\frac{l}{m}, \frac{ln}{mm}, \frac{lnn}{m^3}, \frac{ln^3}{m^4}, \frac{ln^4}{m^5}$ &c. ex hyp. primus terminus eſt ad ſecundum, ut tertius ad quartum, & quintus ad ſextum &c. nec non ſecundus ad tertium, ut quartus ad quintum, & ſextus ad ſeptimum &c. erit etiam ex æquo, primus ad tertium, ut tertius ad quintum, & quintus ad ſeptimum &c. quod docet, primum, tertium, quintum, ſeptimum &c. terminos exemptis reliquis etiam geometricè proportionales eſſe, quorum adeo ſumma per Corollar. VIII. invenitur $\frac{lm}{mm-nn}$. Eodem pacto oſtenditur, ſecundum, quartum, ſextum &c. terminos ſeriem geometricè proportionalium efficere, cujus ſumma $\frac{ln}{mm-nn}$. Igitur differentia harum duarum serierum, seu $\frac{l}{m} - \frac{ln}{mm} + \frac{lnn}{m3} - \frac{ln3}{m4} + \frac{ln4}{m5}$ &c. ∞ $\frac{lm - ln}{mm - nn} \infty \frac{l}{m+n}$, ac propterea quantitas $\frac{l}{m+n}$ in istam seriem vicissim converti potest.

Coroll. 1. In omni Progressione Geometrica descendente (primo termino existente determinato, signisque + & − alternatim se excipientibus) summa seriei limites habet, quos nequit attingere, nedum egredi, qualiscunque statuatur ratio progressionis. Cum enim per hyp. $n > 0$, & $< m$, erit $\frac{l}{m+n} < \frac{l}{m+0} = \frac{l}{m}$; & $> \frac{l}{m+m} = \frac{l}{2m}$, hoc est, valor seriei perpetuo minor est ipso primo termino, & major ejus semisse.

Coroll. 2. Si tamen $m \infty n$, fiet $\frac{l}{m+n} \infty \frac{l}{2m}$, & series $\frac{l}{m} - \frac{ln}{mm} + \frac{lnn}{m3} - \frac{ln3}{m4}$ &c. $\infty \frac{l}{m} - \frac{l}{m} + \frac{l}{m} - \frac{l}{m}$ &c. unde paradoxum fluit non inelegans, quod $\frac{l}{m} - \frac{l}{m} + \frac{l}{m} - \frac{l}{m}$ &c. $\infty \frac{l}{2m}$. Etenim si ultimus seriei terminus signo − affectus concipiatur, termini omnes se mutuò destruere apparebunt, & si signo +, æquari videbuntur ipsi $\frac{l}{m}$, non $\frac{l}{2m}$. Ratio autem paradoxi est, quod continuata divisione ipsius l per $m+m$, residuum divisionis non minuitur, sed perpetuo ipsi l æquale manet; unde quotiens divisionis propriè non est sola series $\frac{l}{m} - \frac{l}{m} + \frac{l}{m} - \frac{l}{m}$ &c. sed $\frac{l}{m} - \frac{l}{m} + \frac{l}{m} - \frac{l}{m}$ &c. + vel $- \frac{l}{2m}$, faciendo scil. fractionem ex residuo & divisore, illamque signo + vel − afficiendo, prout ultimus seriei terminus vicissim − vel + habere fingitur.

XXXVIII. *Fractionem* $\frac{1}{\square : m - n}$ *transmutare in seriem infinitam.*

Quoniam quantitas $\frac{l}{m-n} \infty \frac{l}{m} + \frac{ln}{mm} + \frac{lnn}{m3} + \frac{ln3}{m4}$ &c. per XXXVI. facta utrinque multiplicatione per $\frac{1}{m-n}$, habebitur $\frac{l}{\square : m-n} \infty$ seriei

seriei A, cujus termini singuli de novo in totidem alias series B, C, D, E, F, &c. per eandem XXXVI. Prop. convertantur. Quo facto, serierum istarum termini homologi in unam summam

$$\frac{l}{\square : m-n} \infty A \begin{cases} \frac{l}{mm - mn} \infty \frac{l}{mm} + \frac{ln}{m3} + \frac{lnn}{m4} + \frac{ln3}{m5} + \frac{ln4}{m6} \text{ \&c.} \infty B \\ \frac{ln}{m3 - mmn} \infty \cdot + \frac{ln}{m3} + \frac{lnn}{m4} + \frac{ln3}{m5} + \frac{ln4}{m6} \text{ \&c.} \infty C \\ \frac{lnn}{m4 - m3n} \infty \cdot \quad \cdot + \frac{lnn}{m4} + \frac{ln3}{m5} + \frac{ln4}{m6} \text{ \&c.} \infty D \\ \frac{ln3}{m5 - m4n} \infty \cdot \quad \cdot \quad \cdot \quad \cdot + \frac{ln3}{m5} + \frac{ln4}{m6} \text{ \&c.} \infty E \\ \frac{ln4}{m6 - m5n} \infty \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot + \frac{ln4}{m6} \text{ \&c.} \infty F \\ \text{\&c.} \infty \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \text{ \&c.} \infty \text{ \&c.} \end{cases}$$

$$Z \infty \frac{l}{mm} + \frac{2ln}{m3} + \frac{3lnn}{m4} + \frac{4ln3}{m5} + \frac{5ln4}{m6} \text{ \&c.} \infty \frac{l}{\square : m-n}$$

conflati novam seriem Z constituent, æqualem propterea quantitati propositæ $\frac{l}{\square : m-n}$, mixtamque ex serie numerorum naturalium 1. 2. 3. 4. &c. & quantitatum geometricé progressionalium $\frac{l}{mm}$, $\frac{ln}{m3}$, $\frac{lnn}{m4}$, $\frac{ln3}{m5}$, &c.

Eadem series Z elici quoque potest divisione continua numeratoris l per denominatorem $mm - 2mn + nn$, dicendo: mm in l, habeo $\frac{l}{mm}$, quod ductum in divisorem & subtractum ex dividendo relinquit $+ \frac{2ln}{m} - \frac{lnn}{mm}$; tum porrò mm in $+ \frac{2ln}{m}$, reperio $+ \frac{2ln}{m3}$, quod multiplicatum & subtractum, ut decet, residuum efficit $+ \frac{3lnn}{mm} - \frac{2l3}{m3}$, atque ita ulterius pergendo in infinitum: quo pacto observabitur, post singulas operationes duo membra reliqua manere, sed illa usque & usque minora, tandemque data quavis quantitate propius ad nihilum vergentia.

Idem etiam ostenditur ex lege reciprocorum, resolvendo seriem Z methodo Prop. XIV. in infinitas series geometricas B, C, D, E, F &c. harum enim summæ cum novam progressionem A constituant,

tuant,

tuant, quæ ipsa summam efficit $\frac{l}{mm - 2mn + nn}$, sequitur reciprocè, & hanc quantitatem $\frac{l}{\square : m - n}$ per seriem Z legitimè efferri posse.

XXXIX. *Fractionem* $\frac{1}{\square : m + n}$ *convertere in seriem.*

Si operatio instituatur methodo Propos. præced. eadem, quæ ibi, obtinebitur series, nisi quod termini locorum parium acquirant signum $-$, sic ut habeatur: $\frac{l}{\square : m + n} \infty \frac{l}{mm} - \frac{2ln}{m^3} + \frac{3lnn}{m^4} - \frac{4ln^3}{m^5} + \frac{5ln^4}{m^6} - \frac{6ln^5}{m^7}$ &c.

XL. *Fractionem* $\frac{1}{C : m - n}$, *aut* $\frac{1}{C : m + n}$, *exprimere per seriem.*

Ex analogia operationum præcedentium liquet modus hoc efficiendi; quorsum igitur plura? En operationem:

$\frac{l}{\square : m - n} \infty \frac{l}{mm} + \frac{2ln}{m^3} + \frac{3lnn}{m^4} + \frac{4ln^3}{m^5} + \frac{5ln^4}{m^6}$ &c. per XXXVIII, factaque hinc inde multiplicatione per $\frac{1}{m - n}$,

$$\frac{l}{C : m - n} \infty \left\{ \begin{array}{lllllll} \frac{l}{m^3 - mmn} \infty \frac{l}{m^3} & + \frac{ln}{m^4} & + \frac{lnn}{m^5} & + \frac{ln^3}{m^6} & + \frac{ln^4}{m^7} & \text{\&c.} \\ \frac{2ln}{m^4 - m^3 n} \infty \;\cdot & + \frac{2ln}{m^4} & + \frac{2lnn}{m^5} & + \frac{2ln^3}{m^6} & + \frac{2ln^4}{m^6} & \text{\&c.} \\ \frac{3lnn}{m^5 - m^4 n} \infty \;\cdot & \cdot & + \frac{3lnn}{m^5} & + \frac{3ln^3}{m^6} & + \frac{3ln^4}{m^7} & \text{\&c.} \\ \frac{4ln^3}{m^6 - m^5 n} \infty \;\cdot & \cdot & \cdot & + \frac{4ln^3}{m^6} & + \frac{4ln^4}{m^7} & \text{\&c.} \\ \text{\&c.} \infty \;\cdot & \cdot & \cdot & \cdot & \text{\&c.} & \end{array} \right\} \text{per Prop. XXXVI.}$$

$$\frac{l}{m^3} + \frac{3ln}{m^4} + \frac{6lnn}{m^5} + \frac{10ln^3}{m^6} + \frac{15ln^4}{m^7} \text{ \&c.} \infty \frac{l}{C : m - n}.$$

Eodem pacto habetur $\frac{l}{C : m + n} \infty \frac{l}{m^3} - \frac{3ln}{m^4} + \frac{6lnn}{m^5} - \frac{10ln^3}{m^6} + \frac{15ln^4}{m^7}$ &c.

Conflantur autem termini harum serierum ex ductu terminorum progressionis geometricæ in numeros trigonales 1. 3. 6. 10. 15. &c.

Si quis idem per divisionem continuam consequi desideret, is obser-

observabit, post singulas operationes tria superesse membra, sed ea subinde minora, ultimoque prorsus evanescentia.

Idem etiam regrediendo à serie inventâ patebit, si illa methodo Prop. XIV. in alias resolvatur, &c.

Schol. Haud dissimili operatione reperitur $\frac{l}{QQ : m \pm n} \infty \frac{l}{m^4} \pm \frac{4ln}{m^5} + \frac{10lnn}{m^6} \pm \frac{20ln^3}{m^7}$ &c. ut & $\frac{l}{Ss : m \pm n} \infty \frac{l}{m^5} \pm \frac{5ln}{m^6} + \frac{15lnn}{m^7} \pm \frac{35ln^3}{m^8}$ &c. seriebus mixtis ex geometrica & serie pyramidalium, trianguli-pyramidalium, & ita consequenter in omnibus altioribus, servatâ semper eadem analogiæ ratione, ut non opus sit his diutius immorari.

XLI. *Si proponatur series differentialium, quæ mixta sit ex serie geometrica quantitatum indeterminatarum, & alia quavis serie quantitatum constantium seu coëfficientium, integralia eorum absoluta seriem constituent mixtam ex eadem serie coëfficientium, simili geometrica indeterminatarum, & alia quadam harmonica.*

Patet ex princ. calc. diff. vel summatorii, juxta quæ quantitatis differentialis $nx^m dx$ integrale absolutum reperitur $\frac{n.x^{m+1}}{m+1}$; hinc enim si coëfficientes n sint progressionis cujusvis, & exponentes m progressionis arithmeticæ, h. e. ipsa x^m progr. geometricæ, erunt quoque $m+1$ arithm. adeoque x^{m+1} geometr. & $\frac{1}{m+1}$ harmonicæ progressionis. Ut si proponatur series differentialium $axdx$, bx^3dx, cx^5dx, fx^7dx &c. mixta ex serie quavis a, b, c, f &c. & geometrica xdx, x^3dx, x^5dx, x^7dx &c. erunt eorum integralia $\frac{axx}{2}, \frac{bx^4}{4}, \frac{cx^6}{6}, \frac{fx^8}{8}$ &c. mixta ex eadem serie a, b, c, f &c. geometrica simili xx, x^4, x^6, x^8 &c. & harmonica $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$ &c.

XLII. *Exhibere aream Hyperbolæ inter Asymptotas per seriem infinitam.* Fig. 1.

Mod. 1. *Per Arithm. Infin. Wall.* Esto Hyperbola PCQ, cujus centrum A, asymptotæ AD, AS, applicatæ BC, IO (io), quærendumque sit spatium CBIO (CBio). Sumto autem AB ∞ 1 ∞

Mm BD,

BD, BC $= b$, BI (Bι) $= x$, quæ non sit $>$ AB vel BD, h. e. unitate. Dividatur BI (Bι) in partes aliquot æquales BE, EF, FG, GR, RI (Bε, εφ, φγ, γρ, ρι) quarum numerus sit n, & singulæ dicantur d, sic ut nd sit $= x =$ BI (Bι). Tum circumscribantur (inscribantur) hyperbolæ parallelogramma BK, EL, FM, GN, RO (Bκ, ελ, φμ, γν, ρο) ductis applicatis EK, FL, GM, RN, IO (εκ, φλ, γμ, ρν, ιο) quæ ex natura hyperb. ordine reperiuntur $= \frac{b}{1 \mp d}, \frac{b}{1 \mp 2d}, \frac{b}{1 \mp 3d}, \frac{b}{1 \mp 4d}$ &c. usque ad ultimam $\frac{b}{1 \mp nd}$. Singulis igitur in d ductis, habentur areæ parallelogrammorum, quæ porro in series convertendæ sunt per XXXVI. & XXXVII, ut sequitur:

$$\text{BK(B}\kappa) = \frac{bd}{1 \mp d} = bd \mp bdd + bd^3 \mp bd^4 + bd^5 \mp bd^6 \text{ \&c.}$$

$$\text{EL}(\varepsilon\lambda) = \frac{bd}{1 \mp 2d} = bd \mp 2bdd + 4bd^3 \mp 8bd^4 + 16bd^5 \mp 32bd^6 \text{ \&c.}$$

$$\text{FM}(\varphi\mu) = \frac{bd}{1 \mp 3d} = bd \mp 3bdd + 9bd^3 \mp 27bd^4 + 81bd^5 \mp 243bd^6 \text{ \&c.}$$

$$\text{GN}(\gamma\nu) = \frac{bd}{1 \mp 4d} = bd \mp 4bdd + 16bd^3 \mp 64bd^4 + 256bd^5 \mp 1024bd^6 \text{ \&c.}$$

$$\text{Ult. RO}(\rho o) = \frac{bd}{1 \mp nd} = bd \mp nbdd + nnbd^3 \mp n^3bd^4 + n^4bd^5 \mp n^5bd^6 \text{ \&c.}$$

Harum serierum primi termini æquantur, secundi progrediuntur ut numeri naturales, tertii ut eorundem quadrata, quarti ut cubi, &c hinc posito numero serierum seu parallelogrammorum n infinito (quo quidem casu summa parall-orum seu inscript. seu circumscriptorum ab ipso curvilineo CBIO vel CBιο non differt) summa terminorum primæ seriei perpendicularis erit æqualis, terminorum secundæ dimidia, tertiæ subtripla &c. summæ totidem, hoc est, n terminorum ultimo æqualium, per ea, quæ docet Wallisius in Arithm. Infinit. nosque demonstrabimus alibi: ac propterea summa omnium serierum perpendicularium, i. e. omnium parallelogrammorum, seu area spatii hyperbolici CBIO (CBιο) hac serie exprimetur:

$$\frac{nbd}{1} \mp \frac{nnbdd}{2} + \frac{n^3bd^3}{3} \mp \frac{n^4bd^4}{4} + \frac{n^5bd^5}{5} \mp \frac{n^6bd^6}{6} \text{ \&c.}$$

sive, loco nd substituendo x,

bx

$$\frac{bx}{1} \mp \frac{bxx}{2} + \frac{bx^3}{3} \mp \frac{bx^4}{4} + \frac{bx^5}{5} \mp \frac{bx^6}{6} \text{ \&c.}$$

Mod. 2. *Per Calc. diff. Leibn.* Positis, ut prius, AB $= 1 =$ BD, BC $= b$, & BI (Bι) $= x$, ejusque elemento RI (ρι) $= dx$, erit ex natura hyperbolæ IO (ιο) $= \frac{b}{1 \mp x}$, & elementum spatii hyperbolici RO (ρο) $= \frac{bdx}{1 \mp x} =$ seriei geometricæ $bdx \mp bxdx + bxxdx \mp bx^3dx + bx^4dx$ &c. per XXXVI. & XXXVII; adeoque summa elementorum $\int \frac{bdx}{1 \mp x}$, sive spatium CBIO (CBιο) $= bx \mp \frac{bxx}{2} + \frac{bx^3}{3} \mp \frac{bx^4}{4} + \frac{bx^5}{5}$ &c. eadem series, quæ suprà, mixta sc. ex geometrica & harmonica, per præced. Hæc igitur si summari posset, daretur Hyperbolæ quadratura.

Coroll. 1. Si BI $=$ Bι, dabitur tum summa tum differentia spatiorum CBIO & CBιο per seriem ex geom. & harm. mixtam: cum enim sit ostensum

$$\text{CBIO} = bx + \frac{bxx}{2} + \frac{bx^3}{3} + \frac{bx^4}{4} + \frac{bx^5}{5} + \frac{bx^6}{6} \text{ \&c.}$$

$$\text{CB}\iota o = bx - \frac{bxx}{2} + \frac{bx^3}{3} - \frac{bx^4}{4} + \frac{bx^5}{5} - \frac{bx^6}{6} \text{ \&c.}$$

fiet facta serierum additione & subtractione,

$$\text{CBIO} + \text{CB}\iota o = \frac{2bx}{1} + \frac{2bx^3}{3} + \frac{2bx^5}{5} \text{ \&c.}$$

$$\text{CBIO} - \text{CB}\iota o = \frac{2bxx}{2} + \frac{2bx^4}{4} + \frac{2bx^6}{6} \text{ \&c.}$$

Coroll. 2. Posita BI, $x =$ BA, 1, fit spatium interminatum hyperbolicum PCBAS $= \frac{b}{1} + \frac{b}{2} + \frac{b}{3} + \frac{b}{4} + \frac{b}{5} + \frac{b}{6}$ &c. simplici seriei harmonicæ, quæ cum infinita sit per XVI, arguit & aream hujus spatii talem esse. Conf. Cor. 4. ejusd. Prop.

Cor. 3. Sin & Bι, x, $=$ BD, $1 =$ BC, b, resultat pro spatio CBDQ series harmonica $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ &c. hoc est, subducendo unumquemque terminum signo $-$ affectum à præcedenti, series $\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56}$ &c. cujus termini per saltum excerpti sunt ex serie reciproca trigonalium Q Prop. XV, $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$ &c.

Quòd

Quòd ſi ſtatuatur □ AB, BC vel BD quadruplo minus, np. $\frac{1}{4}$, exhibebitur etiam ſpatium CBDQ per ſeriem prioris ſubquadruplam $\frac{1}{8} + \frac{1}{48} + \frac{1}{120}$ &c. quæ per ſaltum formatur ex ſerie I Propoſ. XVII. Conf. Act. Lipſ. 1682. p. 46.

XLIII. *Invenire aream ſpatii ABEFS (BDφ) comprehenſi Aſymptota hyperbolæ AD, & Curva BEF (ιφ), quæ talis, ut ▭ ſub ejus applicata IE (ιε) & recta conſtante AB, BC vel BD (quæ ſit 1) æquetur ſpatio hyperbolico CBIO (CBιο).* Fig. 2.

Quoniam, poſita BI ꝏ x, ſpatium hyperbolicum CBIO ꝏ $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$ &c. per præced. eadem quoque ſeries denotabit (ob AB vel BC ꝏ 1) longitudinem applicatæ IE, quæ propterea ducta in IR ſeu dx producit $x\,dx + \frac{xx\,dx}{2} + \frac{x^3dx}{3} + \frac{x^4dx}{4} + \frac{x^5dx}{5}$ &c. ꝏ RE, elem. ſpatii BIE. Hujus ſeriei terminos ſummando fit ſpatium BIE ꝏ $\frac{xx}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20} + \frac{x^6}{30}$ &c. ſeriei mixtæ ex geometrica & reciproca trigonalium, quæ poſito inſuper BI, x ꝏ BA, 1, mutatur in ſimplicem trigonalium reciprocam $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}$ &c. cujus ſumma ꝏ 1, per XV. Eſt igitur totum ſpatium ABEFS abſolutè quadrabile, æquale quippe □to AB. Nota hic exemplum Curvæ mechanicæ, ubi quadratura ſpecialis ſuccedit abſque generali; ſimplicis enim ſeriei ſummam dedimus, mixtæ non item.

Eadem ratione oſtendetur ex altero latere ſpatium Bιε ꝏ $\frac{xx}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20}$ &c. totumque ſpatium BDφ ꝏ $\frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20}$ &c.

Coroll. Completis rectangulis CD & BQ, ajo fore curvilineum mechanicum BDφ ꝏ duplo curvilineo hyperbolico CQL, differentiam curvilineorum ABEFS & BDφ ꝏ duplo ſpatio CQH, & ſummam eorundem ꝏ 2 CBDQ; quæ ſic palàm fiunt: Si à ſerie $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$ &c. ſubducatur $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}$ &c. auferendo ſigillatim primum terminum à primo, ſecundum à ſecundo, tertium à tertio, &c. relinquetur $\frac{1}{2} - \frac{1}{6} + \frac{1}{12} -$

$\frac{1}{12} - \frac{1}{20} + \frac{1}{30} - \frac{1}{42}$ &c. ꝏ ſpatio BDφ, ut oſtenſum. Si vero eadem ſeries ex altera ſic tollatur, ut primus ejus terminus dematur ex ſecundo alterius, ſecundus ex tertio, tertius ex quarto &c. orietur $\frac{1}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6}$ &c. ꝏ $\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4}$ &c. $-$ 1 ꝏ (per præced.) duplo ſpat. hyperb. CBDQ $-$ BH ꝏ 2 CBDQ $-$ 2 DL ꝏ 2 CLQ. Ergo BDφ ꝏ 2 CLQ. Igitur cum oſtenſum etiam ſit ABEFS ꝏ 1 ꝏ BH ꝏ 2 DL ꝏ 2 LH, erit ABEFS $-$ BDφ ꝏ 2 LH $-$ 2 CLQ ꝏ 2 CQH; nec non ABEFS + BDφ ꝏ 2 DL + 2 CLQ ꝏ 2 CBDQ. Quæ erant demonſtr.

XLIV. *Invenire aream ſpatii ABKGMT (BDNγK) comprehenſi aſymtota hyperbolæ AD, & Curva KGM (KγN), quæ talis, ut ▭ BIG (Bιγκ) ſub ejus applicata IG (ιγ) & indeterminata BI (Bι), æquetur ſpatio hyperbolico CBIO (CBιο).* Fig. 2.

Quia poſitis omnibus, ut prius, ſpatium hyperb. CBIO ꝏ $x + \frac{xx}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$ &c. per XLII, erit per hyp. facta diviſione per BI ſeu x, recta IG ꝏ $1 + \frac{x}{2} + \frac{xx}{3} + \frac{x^3}{4} + \frac{x^4}{5}$ &c. adeoque RG elem. ſpat. BIGK ꝏ $dx + \frac{x\,dx}{2} + \frac{xx\,dx}{3} + \frac{x^3dx}{4}$ &c. omniaq; RG ſeu ſpatium BIGK ꝏ $\frac{x}{1} + \frac{xx}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \frac{x^5}{25}$ &c. & poſita x ꝏ 1, ſpatium totale ABKGMT ꝏ $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$ &c. ſeriei reciprocæ quadratorum, cujus ſummam etiamnum deſideramus. Conf. Prop. XVII. ſub fin.

Haud diſſimili modo reperitur ex altera parte ſpatium BιγK ꝏ $\frac{x}{1} - \frac{xx}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25}$ &c. ſumtaque x ꝏ 1, totale ſpatium BDNγK & $\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25}$ &c.

Coroll. Spatium ABKGMT duplum eſt ſpatii BDNγK; cum enim ſumma utriusque ſit $\frac{2}{1} + \frac{2}{9} + \frac{2}{25}$ &c. & differentia $\frac{2}{4} + \frac{2}{16} + \frac{2}{36}$ &c. erit utique ſumma ad differentiam, ut $\frac{1}{1} + \frac{1}{9} + \frac{1}{25}$ &c. ad $\frac{1}{4} + \frac{1}{16} + \frac{1}{36}$ &c. h. e. ut 3 ad 1, per XXIV; unde ſpatium unum alterius duplum eſſe neceſſe eſt, ut maximè neutrius abſolutam magnitudinem exploratam habeamus. Vid. Schol. ibid.

XLV. *Exhibere Quadraturam Circuli aut Rectificationem Lineæ Circularis per seriem.* Fig. 3.

In peripheria semicirculi BCD, sumto indefinitè puncto H, demittatur ex illo in radium AB perpendicularis HE; & sit AB $\infty\ 1$, & BE $\infty\ x$, adeoque ex nat. circ. EH $\infty \sqrt{2x-xx}$: quo posito, cum ob simil. Triangul. characteristici LGH & Triangul. HEA, HE sit ad HA, sicut LG vel EF elem. abscissæ BE, ad LH elem. arcus circ. BH, reperitur LH $\infty \frac{dx}{\sqrt{2x-xx}}$, factaque multiplicatione per $\frac{1}{2}$, semissem radii AH, sector HAL seu elem. sectoris HAB $\infty \frac{dx}{2\sqrt{2x-xx}}$. Hæc igitur quantitas, cum absolutè summari nequeat, in seriem convertenda est, sed prius tollenda irrationalitas, quod eo ferè modo fit, quo in Problematis Diophanteis uti vulgo sueverunt. In hunc finem pono $\sqrt{2x-xx}$ $\infty \frac{x}{t}$, seu $2x-xx \infty \frac{xx}{tt}$, ubi quia divisio fieri potest per x, ipsaque non nisi unius dimensionis in æquatione relinquitur, ejus valor in rationalibus prodibit, unde & dx, & per hypoth. $\sqrt{2x-xx}$ seu $\frac{x}{t}$, ipsaque adeo fractio $\frac{dx}{2\sqrt{2x-xx}}$, rationales fient; nempe $x \infty \frac{2tt}{1+tt}$, $dx \infty \frac{4t\,dt}{\square : 1+tt}$, $\sqrt{2x-xx} \infty \frac{x}{t} \infty \frac{2t}{1+tt}$, & deniq; $\frac{dx}{2\sqrt{2x-xx}}$ $\infty \frac{dt}{1+tt}$; hinc fractio in seriem geometricam per XXXVII conversa exhibet $dt - tt\,dt + t^4\,dt - t^6\,dt + t^8\,dt$ &c. Summa igitur elementorum HAL, seu totus sector HAB $\infty\ t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9}$ &c. eoque per semissem radii $\frac{1}{2}$ diviso, arcus BH ∞ $\frac{2t}{1} - \frac{2t^3}{3} + \frac{2t^5}{5} - \frac{2t^7}{7} + \frac{2t^9}{9} - \frac{2t^{11}}{11}$ &c. quæ series mixtæ sunt ex geometrica & harmonica, per XLI, à quarum proin summatione decantatum illud de Circuli Tetragonismo Problema dependet. Nota, ductis ex B & H tangentibus circuli BI, HI, sibi mutuo occurrentibus in I, junctaque HD, quæ radium AC secet in K, fore BI vel IH ∞ AK, utramlibet autem $\infty\ t$. Nam 2. ang. BAI ∞ BAH ∞ AHD + ADH ∞ 2 ADH. Ergo BAI ∞ ADH;

ADH; cumque & ABI & DAK anguli, nec non latera AB & AD æquentur, erit quoque BI ∞ AK. Deinde cum sit per hypoth. 1 ad t, ut $\sqrt{2x-xx}$ ad x; itemque, ob sim. △ DAK & DEH, AD seu 1 ad AK, sicut DE ad EH, hoc est, ex nat. circul. HE ad EB, seu $\sqrt{2x-xx}$ ad x; erit utique $1 . t :: 1 .$ AK, ac proinde AK seu BI $\infty\ t$.

Coroll. 1. Sumta $t \infty 1$, quo casu & BE, x seu $\frac{2tt}{1+tt}$, æquatur BA, 1, fiet quadrans BAC ∞ simplici seriei harmonicæ $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$ &c. ∞ (subducto reapse unoquoque termino signo − affecto à præcedente) $\frac{2}{3} + \frac{2}{35} + \frac{2}{99}$ &c. Hinc quia quadratum radii est ad quadrantem circuli, sicut quadratum diametri ad totum circulum, sequitur si quadratum diametri, h. e. quadratum circulo circumscriptum sit 1, ac proin eidem inscriptum $\frac{1}{2}$, totius circuli aream per modo memoratam seriem expressum iri; adeoque si quadratum circulo inscriptum sit $\frac{1}{4}$, circuli aream fore $\frac{1}{3} + \frac{1}{35} + \frac{1}{99}$ &c. cujus seriei termini per saltum excerpti sunt ex serie H Prop. XVII. Conf. Act. Lipſ. 1682. p. 45.

Coroll. 2. Posita Tangente BI $\infty\ t$, erit arcus, cujus tangens est, $\infty \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$ &c. utpote semissis arcus BH. Confer Act. Lipſ. 1691. pag. 179.

XLVI. *Exhibere generaliter Sectorem cujusvis Sectionis Conicæ ex centro per seriem.* Fig. 4. & 5.

Esto Coni Sectio quæcunque, Hyperbola sive Ellipsis, BCD, cujus centrum A, vertex B, semi-latus transversum AB $\infty\ a$, semi-axis conjugatus AL $\infty\ 1$, adeoque semi-latus rectum $\infty \frac{1}{a}$, & ratio laterum, ut aa ad 1. Ponanturque porrò, abscissa indeterminata BG $\infty\ x$, AG $\infty\ z \infty a \pm x$ ($\pm$ significat + in Hyperbol. & − in Ellipſ. uti $\mp$ vicissim, − in Hyp. & + in Ell.) ejusque elementum FG vel CH $\infty\ dx \infty \pm dz$, ordinata GD $\infty\ y$, ejus elementum DH $\infty\ dy$, & jungens D cum centro recta AD $\infty\ u \infty$ $\sqrt{zz+yy}$. Ducta etiam intelligatur HCI parallela axi, secansque

que curvam in C & rectam AD in I, atque ex C demissa concipiatur in AD perpendicularis CE. Quibus positis, erit primo ex nat. Curv. $aa.\ 1 :: \pm zz \pm aa\,(2ax \pm xx).\ yy$; unde fit $aayy \infty \pm zz \pm aa\,(2ax \pm xx)$, & differentiando $aaydy \infty \pm zdz$, & denique $dy \infty \frac{\pm zdz}{aay} \infty \frac{zdx}{aay}$. Deinde quoniam, ob sim. △ DGA & DHI, DG, y est ad GA, z, sicut DH, dy, ad HI, invenitur HI $\infty \frac{zdy}{y}$, ac proinde CI (HI $\pm$ HC) $\infty \frac{zdy}{y} - dz \infty \frac{zdy - ydz}{y}$. Quare denuò propter △ sim. AGD & IEC, ut AD, u, ad DG, y, sic IC, $\frac{zdy - ydz}{y}$, ad CE; unde reperitur CE $\infty \frac{zdy - ydz}{u}$, quæ ducta in semissem AD, seu $\frac{1}{2}u$, dat aream trianguli elementaris ACD $\infty \frac{zdy - ydz}{2} \infty$ (posito loco dy valore ejus) $\frac{\pm zzdz}{2aay} - \frac{ydz}{2} \infty \frac{\pm zzdz - aayydz}{2aay} \infty$ (substituendo $\pm zz \pm aa$ loco $aayy$) $\frac{\pm zzdz \pm zzdz \pm aadz}{2aay} \infty \frac{\pm adz}{2ay} \infty \frac{adx}{2ay} \infty$ (loco ay surrogando $\sqrt{2ax \pm xx}$) $\frac{adx}{2\sqrt{2ax \pm xx}}$, de qua in seriem convertenda & summanda agitur. Primò autem irrationalitas ex illa tollenda, mediante alia indeterminata, quæ loco x surrogari debet, ut in præc. Pono itaque $\sqrt{2ax \pm xx} \infty \frac{x}{t}$, unde fluit $x \infty \frac{2att}{1 \pm tt}$, & $dx \infty \frac{4atdt}{\square : 1 \pm tt}$, & $\sqrt{2ax \pm xx} \infty \frac{x}{t} \infty \frac{2at}{1 \pm tt}$, & denique $\frac{adx}{2\sqrt{2ax \pm xx}} \infty \frac{adt}{1 \pm tt} \infty$ seriei geom. $adt \pm attdt + at^4dt \pm at^6dt + at^8dt$ &c. per XXXVI & XXXVII. Summa igitur omnium sectorum elementarium ACD, i. e. area totius Sectoris ABCD $\infty at \pm \frac{at^3}{3} + \frac{at^5}{5} \pm \frac{at^7}{7} + \frac{at^9}{9}$ &c. ▭ scil. comprehenso sub a semi-latere transverso & recta, cujus longitudo est $t \pm \frac{t^3}{3} + \frac{t^5}{5} \pm \frac{t^7}{7} + \frac{t^9}{9}$ &c. Unde patet, quo pacto generaliter quadraturæ sectionum Conicarum ad summas serierum ex geomet. & harmon. mixtarum reducantur.

Nota,

Nota, ductis per verticem B & punctum Curvæ D tangentibus BM, DT, sibi mutuo occurrentibus in M, dico fore BM $\infty\ t$. Quoniam enim AG. AB :: AB. AT, per 37. lib. 1. Apoll. ac idcircò convertendo AB. TB :: AG, z. GB, x; nec non (ob sim. △ TBM & CHD) TB. BM :: CH, dx. HD, dy :: (ex æquat. Curvæ different.) $aay.\ z$; erit ex æquo perturbatè AB, a. BM :: $aay.\ x$; unde obtinetur BM $\infty \frac{x}{ay} \infty \frac{x}{\sqrt{2ax \pm xx}}$; adeoque $\sqrt{2ax \pm xx}.\ x :: 1.$ BM: verum per constructionem $\sqrt{2ax \pm xx}.\ x :: 1.\ t$. Ergò omnino BM $\infty\ t$. Conf. Act. Lips. 1691. p. 179.

XLVII. *Dato Numero invenire Logarithmum per seriem.* Fig. 6.

Intelligatur super axe SAσ Curva quædam CBκ, ejus naturæ, ut abscissæ AR, AS (Aρ, Aσ) crescant arithmeticè, dum applicatæ RE, SC ($\rho\varepsilon$, $\sigma\kappa$) crescunt vel decrescunt geometricè, h. e. ut istæ sint ut Numeri, dum illæ sunt ut Logarithmi. Vocabitur hæc Curva *Logarithmica*, cujus hæc est proprietas, ostendente Acut. Leibnitio in Act. Lips. 1684. p. 473. ut Subtangentes ejus omnes AK, RN, $\rho\nu$ sint æquales. Applicetur in A recta AB, & sumto quovis in curva puncto E (ε) ducatur recta EI ($\varepsilon\iota$) parallela axi SA; voceturque AB 1, BI (Bι) x; adeoque AI (Aι) seu RE ($\rho\varepsilon$) $1 \pm x$; nec non AR (Aρ) y, & constans curvæ subtangens b. Dato itaque numero RE ($\rho\varepsilon$) ejus Logarithmus AR (Aρ) sic invenitur. Quoniam ex nat. gen. curvarum, elementum applicatæ EF ($\varepsilon\varphi$) dx, est ad elementum abscissæ FG ($\varphi\gamma$) dy, sicut applicata RE ($\rho\varepsilon$) $1 \pm x$, ad curvæ subtangentem RN ($\rho\nu$) b, habebitur $dy \infty \frac{bdx}{1 \pm x} \infty$ fractione in seriem resoluta per XXXVI & XXXVII) $bdx \pm bxdx + bxxdx \pm bx^3dx + bx^4dx \pm bx^5dx$ &c. ideoque facta summatione, y, hoc est, AR (Aρ) $\infty\ bx \pm \frac{bxx}{2} + \frac{bx^3}{3} \pm \frac{bx^4}{4} + \frac{bx^5}{5} \pm \frac{bx^6}{6}$ &c. quæ insuper in casu speciali BI (Bι) ∞ BA ∞ BD, seu $x \infty 1$, fit $b \pm \frac{b}{2} + \frac{b}{3} \pm \frac{b}{4} + \frac{b}{5} \pm \frac{b}{6}$ &c.

Coroll. 1. Identitas hujus feriei cum illa, quam fupra Prop. XLII. pro fpatio Hyperbolico quadrando reperimus, de mutua dependentia & affinitate inter Hyperbolam & Logarithmos nos admonet, perfpicuumque facit, quod fumtis in utraque Fig. 1. & 6. ipfis BI (Bε) æqualibus fpatium hyperbolicum CBIO (CBεο) æquetur ▭lo fub unitate AB & Logarithmo AR (Aρ). Unde porrò infertur, quòd fumtis utrobique AB, Aε, AD, hoc eft, AB, ρε, σκ continuè proportionalibus, quo cafu ex natura Logarithmicæ Aσ dupla fiet ipfius Aρ, fpatium hyperbolicum CBDQ duplum quoque fit ipfius CBεο, indeque CBεο, οε DQ fpatia futura fint æqualia.

Coroll. 2. Quoniam evidens eft, exiftente BI ∞ AB, h. e. evanefcente AI feu RE, Logarithmum AR reddi infinitum, fequitur & feriem harmonicam Logarithmum hunc exprimentem, $b + \frac{b}{2} + \frac{b}{3} + \frac{b}{4} + \frac{b}{5}$ &c. talem effe; unde denuò veritas Prop. XVI. conftat.

Coroll. 3. Dato quovis Logarithmo putà binarii, determinari poteft ex illo curvæ fubtangens b; cum enim pofita BD ∞ 1 ∞ AB, adeoque AD ∞ σκ ∞ 2, oftenfum fit Aσ Log-um binarii effe $\infty\ b - \frac{b}{2} + \frac{b}{3} - \frac{b}{4}$ &c. $\infty\ b$ in $\overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}$ &c. erit viciffim $b \infty \frac{\text{Log. } 2.}{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ \&c.}$

XLVIII. *Dato Sinu complementi reperire Logarithmum Sinus recti per feriem.* Fig. 6.

In eadem fig. centro A per B defcriptus efto circuli quadrans BHσ, quem producta EI fecet in H, erit AI feu RE finus arcus Hσ, & AR ejus Logarithmus, exiftente vid. radii AB ceu unitatis Logarithmo o. Ponatur finus complementi IH $\infty\ x$, ut fiat finus rectus AI feu RE $\infty \sqrt{1 - xx}$, ejusque elementum EF $\infty \frac{-x\,dx}{\sqrt{1-xx}}$, erit ex nat. gen. curv. EF $\frac{-x\,dx}{\sqrt{1-xx}}$ ad FG, elementum Log-i AR; ut RE, $\sqrt{1-xx}$ ad fubtangentem Logarithmicæ RN quæ fit 1; adeoque FG $\infty \frac{-x\,dx}{1-xx} \infty$ (per XXXVI) $-x\,dx - x^3$

$-x^3\,dx - x^5\,dx - x^7\,dx$ &c. Quare fummando fient omnia FG, feu Log-us AR $\infty - \frac{xx}{2} - \frac{x^4}{4} - \frac{x^6}{6} - \frac{x^8}{8} - \frac{x^{10}}{10}$ &c. negativus fcil. quia numerus ejus RE minor eft unitate AB; at fi fiat pofitivus $\frac{xx}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8}$ &c. hoc eft, fi AR transferatur ex altera parte in Aρ, erit is propriè Logarithmus rectæ ρε, id eft (ex natura Log-orum) tertiæ proportionalis ad ipfum finum RE & radium AB; qui tamen Log-us immediatè quoque reperiri potuiffet ex valore numeri fui ρε $\infty \frac{1}{\sqrt{1-xx}}$.

Idem etiam D. Leibnitius Act. Lipf. 1691. p. 180. eleganter hoc modo:

$$\left.\begin{array}{l}\text{Log. } \overline{1-x} \infty -y \infty -x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} \ \&c. \\ \text{Log. } \overline{1+x} \infty +y \infty +x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \ \&c.\end{array}\right\} \text{per præc.}$$

$$\text{Log. } \overline{1-xx} \infty \text{ (ex nat. log.)}$$

$$\text{Log. } \overline{1-x} + \text{log. } \overline{1+x} \infty -xx - \frac{x^4}{2} - \frac{x^6}{3} \ \&c.$$

$$\text{Log. } \sqrt{1-xx} \infty \tfrac{1}{2} \text{log. } \overline{1-xx} \infty - \frac{xx}{2} - \frac{x^4}{4} - \frac{x^6}{6} \ \&c.$$

Coroll. Pofito finu complementi HI hujus fig. ∞ BI vel Bε fig. 1. æquabitur ▭lum fub Logarithmo finus recti AR & radio AB dimidio exceffui, quo fpatium hyperbolicum CBIO fuperat alterum CBεο. Patet ex Cor. 1. XLII, ubi CBIO − CBεο ferie præfentis dupla expreffum legitur. Cæterùm moneri potuiffet ibi, quod fumta z tertia proportionali ad 1 & x, feu pofita $z \infty xx$, feries illa convertatur in aliam $z + \frac{zz}{2} + \frac{z^3}{3} + \frac{z^4}{4}$ &c. qua quoque fpatium hyperbolicum, putà CBGM, exiftente BG $\infty\ z$ vel xx, innuitur. Hinc enim patet, quòd CBIO − CBεο ∞ CBGM; & CBIO − CBGM feu MGIO ∞ CBεο; adeoque (cum his pofitis AI, $1-x$ fit ad AG, $1-xx$, ficut AB, 1 ad Aε, $1+x$) quòd fumtis AI, AG, AB, Aε utcunque proportionalibus fpatia fegmentis IG, Bε infiftentia femper futura funt æqualia.

XLIX. *Applicatam Curvæ Catenaræ exhibere per seriem.* Fig. 6.

Esto Curva $\mu B \lambda$, quam Catena ab extremitatibus suis liberè suspensa proprio pondere format, dicta *Catenaria*; cujus centrum A, vertex B, axis ABD, parameter AB ∞ 1, abscissa $A\sigma \infty z$, & applicata $\sigma\lambda$ vel $\sigma\mu \infty y$. Constat ex iis, quæ Act. Lips. 1691. p. 274. &c. hac de Curva memoriæ prodita leguntur, elementum applicatæ dy esse $\infty \frac{dz}{\sqrt{zz-1}}$. Hinc ad tollendam surditatem pono $\sqrt{zz-1} \infty t-z$; unde fit $z \infty \frac{tt+1}{2t}$, $dz \infty \frac{tt-1}{2tt} dt$, $\sqrt{zz-1} \infty t-z \infty \frac{tt-1}{2t}$, ac denique $\frac{dz}{\sqrt{zz-1}} (dy) \infty \frac{dt}{t}$. Quam porrò fractionem ut in seriem convertam, facio denominatorem bimembrem, substituendo $1+x$ loco t, & dx loco dt; eritque $\frac{dt}{t}$ seu $dy \infty \frac{dx}{1+x} \infty$ (per XXXVII) $dx - xdx + xxdx - x^3dx + x^4dx$ &c. unde omnia dy seu $y \infty x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. Quoniam autem $z \infty \frac{tt+1}{2t}$, hoc est, $tt \infty 2zt - 1$, & t seu $1+x \infty z + \sqrt{zz-1}$, prodibit $x \infty z - 1 + \sqrt{zz-1} \infty$ (facta $\sigma D \infty \sqrt{zz-1}$) $B\sigma + \sigma D \infty BD$; igitur data $A\sigma$, z dabitur BD, x, indeque $\sigma\lambda$ seu y per seriem.

Coroll. Ex serie inventa collata cum Prop. XLVII. liquet, y esse Logarithmum numeri x; unde data Logarithmica κBC, cujus subtangens ∞ AB ∞ 1, puncta Catenariæ reperire proclive. Cùm enim z, hoc est, $A\sigma$ vel $\sigma\lambda$ ($S\mu$) $\infty \frac{tt+1}{2t} \infty \frac{1}{2}t + \frac{1}{2t}$, patet, abscissis hinc inde Logarithmis æqualibus $A\sigma$ (AS) ordinatam Catenariæ $\sigma\lambda$ ($S\mu$) semissem esse oportere summæ duarum ab AB æquidistantium ordinatarum Logarithmicæ $\sigma\kappa$ & SC, quarum illa ∞ AD ∞t, hæc ex natura Log. $\infty \frac{1}{t}$. Atque in hoc ipso consistit elegantissima hujus Curvæ constructio Leibnitiana, quam videsis in Act. Lips. 1691. p. 277. seqq.

L. *Datis*

L. *Datis latitudine loci alicujus in Curva Loxodromica & angulo Rumbi cum meridiano, exhibere longitudinem loci per seriem.* Fig. 3.

Lineam Rumbicam seu Loxodromicam vocant Nautæ, quam navis secundùm eundem venti Rumbum constanter incedens in superficie globi terr-aquei describit; adeoque curva est, quæ omnes meridianos eodem angulo obliquo intersecat. Incipit hæc in Æquatore, indeque versus alterutrum polorum obliquè recedendo, tandem in ipsum polum, quem infinitis gyris ambit, desinit. Sumto in fig. 3. sinus totus, idemque & radius Æquatoris, AC ∞ 1, BCD meridianus, B & D poli, tangens anguli Rumbici ∞t, H punctum in Loxodromica, ejus latitudo HC, sinus latitudinis AE, & sinus complementi HE qui vocetur z, longitudo verò seu arcus æquatoris inter meridianum loci H & principium Loxodromicæ interceptus dicatur x. His positis, per illa quæ in Act. Lips. 1691. p. 284. ostensa sunt, invenitur elementum longitudinis $dx \infty \frac{-tdz}{z\sqrt{1-zz}}$: ad cujus tentandam reductionem pono primò $z \infty \frac{1}{p}$; unde fit $dz \infty \frac{-dp}{pp}$, $\frac{dz}{z} \infty \frac{-dp}{p}$, $\sqrt{1-zz} \infty \frac{\sqrt{pp-1}}{p}$, & denique $\frac{-tdz}{z\sqrt{1-zz}} (dx) \infty \frac{tdp}{\sqrt{pp-1}}$; porrò quidem memini, ejusdem formæ fuisse elementum Catenariæ in præc. pergo ponere sicut ibi, $\sqrt{pp-1} \infty p-q$, indeque elicio $\frac{tdp}{\sqrt{pp-1}} (dx) \infty \frac{-tdq}{q}$, ac rursum statuendo $q \infty 1-r$ tandem obtineo $\frac{-tdq}{q} (dx) \infty \frac{tdr}{1-r}$; quæ quidem quantitas etiam immediatè elici potuisset ex quantitate $\frac{-tdz}{z\sqrt{1-zz}}$ si statim fecissem $z \infty \frac{2-2r}{2-2r+rr}$: at in tales hypotheses incidere sæpenumerò difficile est, nisi jam usu compertum habeatur, quæ formulæ in quas transformari possint. Nota, $r \infty AC - BI$, excessui nempe radii suprà tangentem semissis complementi latitudinis puncti H; etenim supposita BI $\infty 1-r$, ductaque recta BH, cum similia sint triangula HEB, ABI, erit HE, z, ad EB, $1-\sqrt{1-zz}$, ut AB, 1, ad BI, $1-r$, unde

resultat

resultat $z \infty \frac{2-2r}{2-2r+rr}$, ut oportet. Conversa autem per XXXVI. inventa quantitate $\frac{tdr}{1-r}$ in seriem, habetur $dx \infty tdr + trdr + trrdr + tr^3dr$ &c. & facta summatione $x \infty tr + \frac{trr}{2} + \frac{tr^3}{3} + \frac{tr^4}{4}$ &c. Patet igitur, quomodo ex data tangente semissis complementi latitudinis inveniatur longitudo.

Sciendum autem, elementum longitudinis $\frac{-tdz}{z\sqrt{1-zz}}$ adhuc aliter posse reduci, statuendo nempe $\sqrt{1-zz} \infty y$; hinc enim fit $z \infty \sqrt{1-yy}$, $dz \infty \frac{-ydy}{\sqrt{1-yy}}$, & $\frac{-tdz}{z\sqrt{1-zz}}$ $(dx) \infty \frac{tdy}{1-yy} \infty$ (per XXXVI) $tdy + tyydy + ty^4dy + ty^6dy$ &c. ac deniq; omnia dx seu $x \infty ty + \frac{ty^3}{3} + \frac{ty^5}{5} + \frac{ty^7}{7}$ &c. ubi perspicuum est, y seu $\sqrt{1-zz} \infty$ AE sinui recto arcus HC; unde constat ratio definiendi etiam quæsitum ex sinu recto latitudinis, quemadmodum fecit Dn. Leibnitius Act. Lips. 1691. p. 181. Et patet, si in calculo, per quem ad initio memoratam æquationem $dx \infty \frac{-tdz}{z\sqrt{1-zz}}$ perveni, loco quantitatis indeterminatæ ipsum siuum rectum AE præ sinu complementi HE selegissem, me statim ad alteram æquationem immediatè in seriem convertibilem $dx \infty \frac{tdy}{1-yy}$ perventurum fuisse. Cæterum ex eo, quod duæ inventæ series eandem quantitatem x denotant, obiter concludimus, quòd si in circulo sinus cujuslibet arcus AE dicatur y, & AC $-$ BI excessus radii suprà tangentem semissis complementi vocetur r, perpetuò futurum sit $y + \frac{y^3}{3} + \frac{y^5}{5}$ &c. $\infty\ r + \frac{rr}{2} + \frac{r^3}{3}$ &c. Notamus etiam, si locus H sit in ipso polo, quo casu $r \infty 1 \infty y$, fore $x \infty t + \frac{t}{2} + \frac{t}{3} + \frac{t}{4}$ &c. vel $\infty\ t + \frac{t}{3} + \frac{t}{5}$ &c. quarum serierum summæ cum sint infinitæ per XVI, docent longitudinem loci H quoque infinitam esse, adeoque, quod dixi, curvam loxodromicam infinitis polum gyris ambire, priusquam in ipsum incidat.

Coroll.

Coroll. 1. Si in eadem Loxodromica præter locum H alius sit locus notæ latitudinis, cujus sinus rectus $\infty\ v$, & excessus radii suprà tangentem semissis complementi $\infty\ s$, erit similiter ejus longitudo $\infty\ t$ in $v + \frac{v^3}{3} + \frac{v^5}{5}$ &c. vel $\infty\ t$ in $s + \frac{ss}{2} + \frac{s^3}{3}$ &c. adeoque differentia longitudinum utriusque loci erit utriusque seriei differentia, np. t in $\frac{y \div v}{1} + \frac{y^3 \div v^3}{3} + \frac{y^5 \div v^5}{5}$ &c. vel, t in $\frac{r \div s}{1} + \frac{r^2 \div s^2}{2} + \frac{r^3 \div s^3}{3}$ &c. Hinc si in alia quadam Loxodromica duo concipiantur loca latitudine cum prioribus convenientia, erunt, manentibus y & v, vel r & s iisdem, differentiæ longitudinum ut tangentes angulorum, quos Rumbi faciunt ad meridianos. Vid. Act. Lips. 1691. p. 182. & 285.

Coroll. 2. Ex collatione harum serierum cum seriebus Propp. XLII. XLVI. & XLVII. liquet Problematis convenientia cum quadratura Hyperbolæ & Logarithmis. Speciatim notamus, quòd existente subtangente Logarithmicæ $\infty\ t$, quæsita longitudo puncti H sit ipse Logarithmus rectæ $1-r$ seu BI, ut patet ex XLVII; vel etiam (cum D. Leibnitio loc. cit.) semissis Log-i quantitatis $\frac{1+y}{1-y}$ seu $\frac{DE}{EB}$, quod sic ostenditur:

$$\left.\begin{array}{l} \text{Log.}\,\overline{1+y} \infty + ty - \frac{tyy}{2} + \frac{ty^3}{3} - \frac{ty^4}{4} + \frac{ty^5}{5} - \frac{ty^6}{6}\ \text{\&c.} \\ \text{Log.}\,\overline{1-y} \infty - ty - \frac{tyy}{2} - \frac{ty^3}{3} - \frac{ty^4}{4} - \frac{ty^5}{5} - \frac{ty^6}{6}\ \text{\&c.} \end{array}\right\} \text{per XLVII.}$$

$$\text{Log.}\,\frac{\overline{1+y}}{1-y} \infty$$

$$\text{Log.}\,\overline{1+y} - \log.\,\overline{1-y} \infty 2ty \quad + \frac{2ty^3}{3} \quad + \frac{2ty^5}{5}\ \text{\&c.}$$

Coroll. 3. Data longitudine & latitudine loci, dabitur angulus Rumbi cum meridiano; cum enim $x \infty t$ in $r + \frac{rr}{2} + \frac{r^3}{3}$ &c. $\infty\ t$ in $\overline{y + \frac{y^3}{3} + \frac{y^5}{5}}$ &c. erit $t\,.\,1 :: x\,.\,r + \frac{rr}{2} + \frac{r^3}{3}$ &c. vel $y + \frac{y^3}{3} + \frac{y^5}{5}$ &c.

&c. id eſt, tangens anguli quæſiti ad ſinum totum, ut arcus longitudinis ad Log-um BI, vel ſemiſſem Log-i $\frac{DE}{EB}$; adeoque per Coroll. 1. hujus, ut differentia duarum longitudinum ad differentiam duorum Log-orum BI, vel ſemi-differentiam duorum $\frac{DE}{EB}$. Intellige hic Logarithmos acceptos in Curva, cujus ſubtangens ∞ radio ∞ 1. Nota, ſi deſideretur angulus Loxodromicæ, quæ non niſi poſt unam pluresve integras revolutiones in datum locum perducat, augendus eſt arcus differentiæ longitudinum integra peripheria æquatoris ejusve multiplo.

Schol. Ex hactenus dictis expeditus habetur modus conſtruendi *Scalam* quandam *Loxodromicam*: Eſto in Fig. 6. BM σ circumferentia æquatoris in gradus ſuos & graduum minutias diviſa; hæc extendatur in rectam AS axem Logarithmicæ CB $\varkappa$, ejuſque diviſionibus ordine ab A adſcribantur gradus longitudinum: tum ſumto indefinitè in circumferentia hac puncto M, biſectoque arcu M σ per rectam AT occurrentem tangenti $\sigma\varkappa$ in T, ducatur ex T recta TE axi AS parallela, ſecans Logarithmicam in E; ac denique ex E demittatur in axem perpendicularis ER; punctoque R adſcribatur numerus graduum in arcu BM: ſic habebuntur etiam gradus latitudinum; parataque erit Scala Loxodromica, quæ primario inſerviet Rumbo, cujus anguli tangens æquatur ſub tangenti Logarithmicæ. Numeri enim graduum cujusvis datæ latitudinis in ſcala ſtatim à latere aſpectui offerunt reſpondentes longitudinis gradus. Eadem tamen etiam cuilibet Rumbo prodeſſe poterit, ſi fiat per Coroll. 1. hujus, ut ſubtangens Logarithmicæ, è qua ſcala conſtructa eſt, ad anguli Rumbici tangentem, ſic longitudo vel differentia longitudinum per ſcalam inventa ad longitudinem vel differentiam longitudinum quæſitam: adeo ut ſcala ejusmodi in uſum nautarum circino proportionis inſculpta, & lineæ partium æqualium, quæ longitudinum gradus repræſentarent, juxta poſita inſtrumentum foret omnium forſan, quæ Naturæ hactenus tractarunt, compendioſiſſimum & utiliſſimum. Sed de his ſatis.

Antequam

Antequam pergamus, Lector advertere poteſt, quòd hucuſque in differentialium ſummatione pro quovis elemento ſemper ejus integrale purum ſeu abſolutum ſubſtituimus, velut x *pro* dx, $\frac{xx}{2}$ *pro* xdx *&c. At ſcire ipſum volumus, hoc minimè eſſe perpetuum; quanquam enim una eademque quantitas* x *non niſi unum habeat differentiale* dx, *idem tamen differentiale* dx *infinita habet integralia, unum quidem purum* x, *reliqua admiſſione quantitatum conſtantium affecta* x + a, x — b &c. *quorum in ſummationis negotio pro re nata nunc hoc nunc illud ſeligendum eſt, neque adeo ſine præſenti hallucinationis periculo indiſcriminatim ſemper purum adſumi poteſt. Reſtat itaque, ut ad vitandum ſcopulum, quem communem ferè eſſe video omnibus iis, qui calculum hunc incautius tractant, ſubjiciamus adhuc ejus rei exemplum in uno alterove Problemate, è cujus enodatione Lectori conſtare poſſit, undenam & quibus criteriis dignoſcatur, quid pro quovis ſemper elemento ſummando ſubſtitui conveniat.*

L. I. *Exhibere longitudinem Curvæ Parabolicæ per ſeriem.* Fig. 4.

Fingamus BCD Curvam eſſe Parabolam, cujus vertex B, axis BG, latus rectum $\infty\, a$, abſciſſa BG $\infty\, x$, applicata GD $\infty\, y$, ipſa BCD curva $\infty\, s$; proinde elem. FG vel CH $\infty\, dx$, DH $\infty\, dy$, & CD $(\sqrt{dx^2 + dy^2}) \infty\, ds$. Erit ex natura Curvæ $ax \infty\, yy$; hinc differentiando $adx \infty\, 2ydy$, quadrandoque $aadx^2$ $(aads^2 - aady^2) \infty\, 4yydy^2$, & facta transpoſitione $aads^2 \infty\, aady^2 + 4yydy^2$, extractaque tandem radice $ads \infty\, dy\sqrt{aa + 4yy}$, quæ quantitas eſt, de qua ſummanda agitur. Ad ſurditatem primò eliminandam pono $\sqrt{aa + 4yy} \infty\, z - 2y$, fiet $aa \infty\, zz - 4zy$, & $y \infty \frac{zz - aa}{4z}$; hinc $dy \infty \frac{zz + aa}{4zz}dz$, nec non $\sqrt{aa + 4yy}$ $(z - 2y) \infty \frac{zz + aa}{2z}$, adeoque $dy\sqrt{aa + 4yy}$ $(ads) \infty \frac{z^4 + 2aazz + a^4}{8z^3}dz \infty$ (membris ſeparatim poſitis) $\frac{zdz}{8} + \frac{aadz}{4z} + \frac{a^4dz}{8z^3}$, de quorum nunc ſummis diſpiciendum. Hunc in finem conſidero relationem, quam habet aſſumta litera indeterminata z ad ordinatas Curvæ noſtræ, eamque ex facta hyp. $\sqrt{aa + 4yy} \infty\, z - 2y$ cognoſco talem eſſe, ut exiſtente $y \infty\, 0$, z non pariter evaneſcat, ſed

Oo fit

ſit ∞a, & quod creſcente y eò fortius creſcere debeat z; quapropter extenſa concipiatur ipſa z in recta DB fig. 3. à puncto D, & ſit prima DA, quæ naſcenti y reſpondet, ∞a, ultimaque z, quæ reſpondet ultimæ y ſeu applicatæ GD fig. 4. eſto DE. Tum fluere intelligatur ab A ad E indefinita recta AK vel EH, æqualis ubique $\frac{zz}{16}$ (integrali ſcil. puro primi membri $\frac{zdz}{8}$) minimaque adeo in A & $\infty \frac{aa}{16}$; ſic ipſum fluentis lineæ incrementum fiet $\frac{zdz}{8}$, & omnia incrementa quæ capit linea, dum ex A movetur in E, repræſentabunt omnia $\frac{zdz}{8}$, quæ ordinatis y à minima (o) ad ultimam (GD) ordine reſpondent, h. e. quæ pertinent ad Curvæ Parabolicæ portionem rectificandam BD (fig. 4.) Conſtituunt autem omnia illa incrementa, ut liquet, non integram EH ($\frac{zz}{16}$) ſed exceſſum tantum ejus ſuprà rectam AK ($\frac{aa}{16}$) hoc eſt, EH — AK ſeu $\frac{zz-aa}{16}$. Integrale igitur primi membri $\frac{zdz}{8}$, quod huc quadrat, eſt $\frac{zz-aa}{16}$. Similiter pro integrando tertio membro $\frac{a^4dz}{8z^3}$, fingo z extendi in recta AD fig. 1. à puncto A, primamque quæ naſcenti y reſpondet eſſe AB ∞a, & quæ reſpondet ultimæ, AD; hinc fluere concipio à B verſus D quantitatem $\frac{a^4}{16zz}$, ceu integrale purum ipſius $\frac{a^4dz}{8z^3}$, putà rectam BC vel DQ, quæ proin maxima erit in B & $\infty \frac{aa}{16}$, indeque verſus D decreſcet; decrementa itaque, quæ patitur linea BC quouſque pervenit in DQ, denotabunt omnia elementa $\frac{a^4dz}{8z^3}$, quæ portioni Curvæ Parabolicæ BD (fig. 4.) reſpondent: ſed omnia illa decrementa, ut apparet, non efficiunt rectam DQ ſeu $\frac{a^4}{16zz}$, verum potius BC — DQ ſeu $\frac{aa}{16} - \frac{a^4}{16zz}$; quapropter integrale tertii membri $\frac{a^4dz}{8z^3}$ huc pertinens

∞aa

$\infty \frac{aa}{16} - \frac{a^4}{16zz}$, ſummaque adeo primi & tertii ($S\frac{zdz}{8} + S\frac{a^4dz}{8z^3}$) $\infty \frac{zz-aa}{16} + \frac{aa}{16} - \frac{a^4}{16zz} \infty \frac{z^4-a^4}{16zz}$.

Reſtat intermedium adhuc membrum expediendum $\frac{aadz}{4z}$. Hoc cum abſolutè ſummari nequeat, in ſeriem converto, ponendo prius $z \infty a + t$, ut denominator fiat bimembris; hinc enim fit $\frac{aadz}{4z} \infty \frac{aadt}{4a+4t}$, ∞ (per XXXVII) $\frac{adt}{4} - \frac{tdt}{4} + \frac{ttdt}{4a} - \frac{t^3dt}{4aa}$ &c. & facta ſummatione, $S\frac{aadz}{4z} \infty \frac{at}{4\cdot1} - \frac{tt}{4\cdot2} + \frac{t^3}{4\cdot3a} - \frac{t^4}{4\cdot4aa} + \frac{t^5}{4\cdot5a^3}$ &c. Nota, quod hic pro quolibet ſeriei termino ſubſtituam ejus integrale purum, quoniam ex æquatione $z \infty a + t$ colligo, quod exiſtente $z \infty a$ (hoc eſt $y \infty o$) ipſa t, ut & quantitates fluentes omnes, $\frac{at}{4\cdot1}$, $\frac{tt}{4\cdot2}$, $\frac{t^3}{4\cdot3a}$ &c. quoque ſint ∞ o, id eſt, quod hæ à o fluere ſeu incrementa ſumere occipiant; hinc enim manifeſtè liquet, omnia ipſarum crementa, nempe omnia $\frac{adt}{4}$, $\frac{tdt}{4}$ &c. ipſis quantitatibus ultimis $\frac{at}{4}$, $\frac{tt}{4\cdot2}$ &c. æqualia fore. Quod idem quoque, ſi quis examinet, in omnibus præcedentium Propp. exemplis contingere obſervabit, indeque concludet, rectè à nobis factum, quod ibidem inter ſummandum pura ſemper integralia aſſumſerimus, tametſi ejus rei rationem disertè non adjecerimus. Sed revertamur ad propoſitum: Inventa ſumma medii membri $\frac{aadz}{4z}$, ſi reliquorum ſummæ ſuprà repertæ adjiciantur, emergit ſumma omnium $S\frac{zdz}{8} + S\frac{aadz}{4z} + S\frac{a^4dz}{8z^3}$, hoc eſt, $as \infty \frac{z^4-a^4}{16zz} + \frac{at}{4\cdot1} - \frac{tt}{4\cdot2} + \frac{t^3}{4\cdot3a} - \frac{t^4}{4\cdot4aa}$ &c. & facta diviſione per a, longitudo Curvæ s ſeu BD $\infty \frac{z^4-a^4}{16azz} + \frac{t}{4\cdot1} - \frac{tt}{4\cdot2a} + \frac{t^3}{4\cdot3aa} - \frac{t^4}{4\cdot4a^3}$ &c. quæ denique poſita $a \infty t \infty 4$, & $z \infty a + t \infty 8$, fit $\frac{15}{16} + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ &c. unde cum ſit hoc caſu $y \infty \frac{zz-aa}{4z} \infty \frac{3}{2}$, & x

∞yy

$\infty \frac{yy}{a} \infty \frac{9}{16}$, sequitur, quod existente latere recto Parabolæ 4, & abscissa BG $\frac{9}{16}$, aut applicata GD $\frac{3}{2}$, longitudo Curvæ Parabolicæ BD æquetur $\frac{15}{16} + \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ &c.

Coroll. Ex serie collata cum XLII. Curvam Parabolicam cum Spatio Hyperbolico inter Asymptotas comparandi modus innotescit. Sufficit monuisse.

LII. *Rectificare Curvam Logarithmicam per seriem & aliter.* Fig. 6.

Insistat axi SAσ Curva Logarithmica CBκ, cujus ordinata AB ∞ 1, subtangens AK $\infty\, b$, alia quævis applicata RE ($\rho\varepsilon$) $\infty\, z$, ejusque elementum EF ($\varepsilon\varphi$) $\infty\, dz$; quæritur rectificatio portionis Curvæ BE (Bε). Quoniam per XLVII. elementum abscissæ AR (Aρ) nempe FG ($\varphi\gamma$) $\infty \frac{bdz}{z}$, erit EG^2 (EF^2 + FG^2) $\infty\, dz^2 + \frac{bbdz^2}{zz} \infty \frac{zz+bb}{zz} dz^2$, indeque elementum curvæ EG ($\varepsilon\gamma$) $\infty \frac{dz\sqrt{zz+bb}}{z} \infty$ (terminis fractionis per $\sqrt{zz+bb}$ æque-multiplicatis) $\frac{zzdz+bbdz}{z\sqrt{zz+bb}} \infty \frac{zdz}{\sqrt{zz+bb}} + \frac{bbdz}{z\sqrt{zz+bb}}$, de quorum summatione hic quæritur. Prioris membri integrale purum est $\sqrt{zz+bb}$, quod (ob primam $z \infty$ AB ∞ 1) inde à $\sqrt{1+bb}$ decrescere (crescere) intelligitur ad usque $\sqrt{zz+bb}$; adeo ut omnia ejus decrementa (incrementa) huc quadrantia, seu $S\frac{zdz}{\sqrt{zz+bb}}$ sint $\infty \sqrt{1+bb} - \sqrt{zz+bb}$ ($\sqrt{zz+bb} - \sqrt{1+bb}$) hoc est, æqualia differentiæ duarum in B & E (ε) tangentium rectarum BK & EN ($\varepsilon\nu$). Posterioris membri $\frac{bbdz}{z\sqrt{zz+bb}}$ integrale, quoniam ita planum non est, prævia reductione investigare conor, eaque simili huic, qua suprà Prop. L. pro Curva Loxodromica fui usus, cum in elementis analogiam quandam observem. Pono itaque primò $z \infty \frac{bb}{p}$, eoque mediante transformo $\frac{bbdz}{z\sqrt{zz+bb}}$ in $\frac{-bdp}{\sqrt{bb+pp}}$; deinde facio $\sqrt{bb+pp} \infty\, p+q$, sive $p \infty \frac{bb-qq}{2q}$; inde-

indeque elicio $\frac{-bdp}{\sqrt{bb+pp}}$ $\left(\frac{bbdz}{z\sqrt{zz+bb}}\right) \infty \frac{bdq}{q}$, quod per XLVII. elementum esse cognosco abscissæ cujusdam in Logarithmica, quam tandem ita determino: Quoniam $p \infty \frac{bb-qq}{2q}$, & $z \infty \frac{bb}{p}$, fiet $z \infty \frac{2bbq}{bb-qq}$, sicut vicissim $q \infty \frac{-bb+b\sqrt{zz+bb}}{z}$; & quia prima $z \infty$ AB ∞ 1, erit quæ huic respondet prima $q \infty -bb + b\sqrt{1+bb}$. Pro constructione, abscindo in tangente BK partem Kα ∞ KA, in ordinata AB partem BV ∞ Bα, & in V statuo VX parallelam ipsi AK; pari modo in tangente $\nu\varepsilon$ (idem imaginatione supple in NE) sumo νr ∞ $\nu\rho$, hinc εv ∞ εr, & duco vx parallelam $\nu\rho$; quo pacto constat fore VX ∞ primæ $q \infty -bb + b\sqrt{1+bb}$, & vx ∞ ultimæ $q \infty \frac{-bb+b\sqrt{zz+bb}}{z}$. Quocirca si ambæ VX & vx, vel etiam loco harum sola quarta proportionalis ad VX, vx & AB (quæ sit SC vel $\sigma\kappa$) applicetur Logarithmicæ, erit intercepta applicatis VX & vx axis portio, vel etiam ipsis AB, SC ($\sigma\kappa$) interjecta portio AS (Aσ) [ex natura enim curvæ æqualis utrisque intercipietur] $\infty\, S\frac{bdq}{q}$, id est, omnibus $\frac{bdq}{q}$, seu omnibus $\frac{bbdz}{z\sqrt{zz+bb}}$ pro portione curvæ BE (Bε) rectificanda inservientibus. Et quoniam posita differentia inter AB & SC ($\sigma\kappa$) $\infty\, x$, resegmentum axis AS (Aσ) $\infty\, bx \mp \frac{bxx}{2} + \frac{bx^3}{3} \mp$ $\frac{bbdz}{z\sqrt{zz+bb}}$ summa etiam per seriem reperta. Addicis itaque amborum summis fient omnia EG ($\varepsilon\gamma$) seu longitudo curvæ BE (Bε) $\infty \sqrt{1+bb} = \sqrt{zz+bb} + bx \mp \frac{bxx}{2} + \frac{bx^3}{3} \mp \frac{bx^4}{4}$ &c. ∞ differentiæ tangentium BK & EN ($\varepsilon\nu$) unà cum resegmento axis AS (Aσ).

Cum non omnes quantitates surdæ, nedum transcendentes, differentialibus admixtæ præcedentibus modis in rationales transformari, inque series converti possint, ad alia subinde nobis artificia recurrendum est ad obtinendum propositum, inter quæ ob universalitatem suam eminent Interpola-

tiones

tiones Walliſianæ, *vel* Exaltatio binomii ad poteſtatem indefinitam, *vel* Aſſumtio ſeriei fictæ inſtar quæſitæ, *aut conſimilia ſubſidia alia, quorum pro re nata nunc unum nunc plura in uſum verti queunt. Nos pauca eorum ſpecimina poſt generalia nonnulla in uno alterove exemplo ſubjungemus.*

LIII. *Quantitatem quamcunque ſurdam vel irrationalem in ſeriem infinitam rationalium convertere per interpolationes Walliſianas.*

Reducatur quantitas rationalis, cujus poteſtas fracta ſive radix aut latus quæritur, ad fractionem hujus formæ $\frac{l}{m-n}$ (ponendo $m < n$). Hujus fractionis poteſtates integræ, prima, ſecunda, tertia, &c. convertantur ope diviſionis continuæ in totidem ſeries, per XXXVI, uſque ad XL, Propp. hoc pacto:

Exp.	*Poteſt.*
0	$1 \propto 1 + 0 + 0 + 0 + 0$ &c.
1	$\frac{l}{m-n} \propto \frac{l}{m} + \frac{ln}{mm} + \frac{lnn}{m3} + \frac{ln3}{m4} + \frac{ln4}{m5}$ &c.
2	$\frac{ll}{Q:m-n} \propto \frac{ll}{mm} + \frac{2lln}{m3} + \frac{3llnn}{m4} + \frac{4lln3}{m5} + \frac{5lln4}{m6}$ &c.
3	$\frac{l3}{C:m-n} \propto \frac{l3}{m3} + \frac{3l3n}{m4} + \frac{6l3nn}{m5} + \frac{10l3n3}{m6} + \frac{15l3n4}{m7}$ &c.
4	$\frac{l4}{Bq:m-n} \propto \frac{l4}{m4} + \frac{4l4n}{m5} + \frac{10l4nn}{m6} + \frac{20l4n3}{m7} + \frac{35l4n4}{m8}$ &c.

In his ſeriebus obſervabis, coëfficientes primorum terminorum conſtituere unitates, coëfficientes ſecundorum numeros laterales, tertiorum trigonales, quartorum pyramidales, & ſic porrò; terminos verò puros ordine oriri ex ductu fractionis $\frac{l}{m}$ (ad poteſtatem elevatæ ſimilem ei ad quam elevanda fractio $\frac{l}{m-n}$) in $1, \frac{n}{m}, \frac{nn}{mm}, \frac{n3}{m3}$, &c. Hinc ad inveniendas poteſtates intermedias ſive radices (ceu media quædam geometrica, quorum exponentes ſunt arithmeticè medii inter exponentes integrorum) numeri terminorum figurati tantum ſunt interpolandi juxta doctrinam Wallisii Prop. 172. ſeqq. Arithm. Infin. Eſt vero, poſito exponente vel indice poteſtatis p, generalis character lateralium quoque p, trigonalium $\frac{p.\overline{p+1}}{1.2}$, pyramidalium $\frac{p.\overline{p+1}.\overline{p+2}}{1.2.3}$, &c. ut ibid. docetur Prop. 182. Quare ſi p interpreteris per $\frac{1}{2}$, invenies poteſtatem dimidiam quantitatis $\frac{l}{m-n}$, nempe $\sqrt{\frac{l}{m-n}} \propto \sqrt{\frac{l}{m}}$ in $\overline{1 + \frac{1n}{2m} + \frac{1.3nn}{2.4mm} + \frac{1.3.5n3}{2.4.6m3} + \frac{1.3.5.7n4}{2.4.6.8m4} + \text{\&c.}}$ ſi p explices per $\frac{1}{3}$, habebis trientem poteſtatis ſeu $\sqrt{C.\frac{l}{m-n}} \propto \sqrt{C.\frac{l}{m}}$ in $\overline{1 + \frac{1n}{3m} + \frac{1.4nn}{3.6mm} + \frac{1.4.7n3}{3.6.9m3} + \frac{1.4.7.10n4}{3.6.9.12m4} + \text{\&c.}}$ ſi per $\frac{3}{2}$, obtinebis ſeſquialteram poteſtatem ſeu $\frac{l}{m-n}\sqrt{\frac{l}{m-n}} \propto \frac{l}{m}\sqrt{\frac{l}{m}}$ in $\overline{1 + \frac{3n}{2m} + \frac{3.5nn}{2.4mm} + \frac{3.5.7n3}{2.4.6m3} + \frac{3.5.7.9n4}{2.4.6.8m4} + \text{\&c.}}$ &c.

Coroll. Quoniam poſitis l, m & n æqualibus inter ſe, fit quantitas $\frac{l}{m-n} \propto \frac{l}{0} \propto \infty$, ſeries autem prædictæ abeunt in ſeries purorum coëfficientium $1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6}$ &c. $1 + \frac{1}{3} + \frac{1.4}{3.6} + \frac{1.4.7}{3.6.9}$ &c. $1 + \frac{3}{2} + \frac{3.5}{2.4} + \frac{3.5.7}{2.4.6}$ &c. colligimus, ſeries ejusmodi natas ex ductu continuo fractionum, quarum numeratores & denominatores in progreſſ. arithm. per differentias primo denominatori æquales insurgunt, ſummas fundere infinitas; quod apertius ita conſtabit: Minue numeratores, eosque æquales conſtitue denominatoribus ſingulos ſingulis, nempe ſecundum numeratorem primo denominatori, tertium ſecundo, 4.um 3tio, & ita deinceps; ſic enim ex. gr. loco primæ ſeriei habebis $1 + \frac{1}{2} + \frac{1.2}{2.4} + \frac{1.2.4}{2.4.6} + \frac{1.2.4.6}{2.4.6.8}$ &c. $\propto$ (perimentibus ſe mutuo dictis numeris) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}$ &c. $\propto \infty$, per Cor. 2. XVI, unde fortius altera $1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6}$ &c. ob numeratores majores infinita erit. Cæterùm poſtremus terminus cujusque ſeriei nunc nullus eſt nunc infinitus, prout exponens poteſtatis p, vel prima ſeriei fractio, unitate minor eſt majorve: Sic ultimus

ultimus terminus primæ seriei $\frac{1.3.5.7}{2.4.6.8}$ &c. nullus est; nam si quantus esset, etiam hic foret quantus $\frac{2.4.6.8}{3.5.7.9}$ &c. utpote cujus singuli factores singulis factoribus præcedentis termini ordine sumtis sunt majores; quare & utriusque productum quantum foret, np. $\frac{1.3.5.7}{2.4.6.8}$ in $\frac{2.4.6.8}{3.5.7.9}$ &c. ꝏ (permistis alternatim utriusque factoribus) $\frac{1.2.3.4.5 \ldots \infty-1}{2.3.4.5.6 \ldots \infty}$ ꝏ (ob numeratores omnes primum sequentes, & denominatores ultimum præcedentes se mutuò perimentes) $\frac{1}{\infty}$ ꝏ 0, quod absurdum. Ultimus contra terminus tertiæ seriei $\frac{3.5.7.9}{2.4.6.8}$ &c. infinitus est; nam si finitus esset, etiam hic foret finitus $\frac{4.6.8.10}{3.5.7.9}$ &c. utpote cujus singuli factores singulis illius sunt minores; quocirca & utriusque productum finitum foret, nempe $\frac{3.5.7}{2.4.6}$ &c. in $\frac{4.6.8}{3.5.7}$ &c. ꝏ $\frac{3.4.5.6 \ldots \infty}{2.3.4.5 \ldots \infty-1}$ ꝏ (destruentibus se mutuo numeratoribus qui ultimum præcedunt, & denominatoribus qui primum sequuntur) $\frac{\infty}{2}$ ꝏ ∞, quod pariter absurdum.

L I V. *Idem præstare per exaltationem binomii ad potestatem indefinitam.*

Quantitas rationalis, cujus potestas per seriem desideratur, sit expressa per binomium $1+n$ (ponendo $1 > n$). Hujus binomii potestas indefinita p, ut jam passim inter Geometras notum, per seriem exprimitur $1+\frac{p}{1}n+\frac{p.\overline{p-1}}{1.2}nn+\frac{p.\overline{p-1}.\overline{p-2}}{1.2.3}n^3+\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{1.2.3.4}n^4$ + &c. ubi perspicuum est, quod quotiescunque exponens potestatis p est numerus integer & positivus, series necessariò aliquando abrumpetur; quandoquidem in continuatione ulteriori coëfficientium $p.\overline{p-1}.\overline{p-2}$. &c. necessariò tandem devenietur ad $p-p$ ꝏ 0; quod proin illum terminum & ab illo deinceps omnes evanescere facit. Sed quoties p numerus fractus est aut negativus, coëfficientes nunquam in nihilum abibunt, ac ideò series in infinitum excurret: qua ratione habetur ex. gr. $\sqrt{1+n}$ (ubi p valet $\frac{1}{2}$) ꝏ

1 +

$1+\frac{1}{2}n-\frac{1}{2.4}nn+\frac{1.3}{2.4.6}n^3-\frac{1.3.5}{2.4.6.8}n^4+$ &c. & $\sqrt{C.\,1+n}$ (ubi p valet $\frac{1}{3}$) ꝏ $1+\frac{1}{3}n-\frac{2}{3.6}nn+\frac{2.5}{3.6.9}n^3-\frac{2.5.8}{3.6.9.12}n^4$ &c. & $\frac{1}{\sqrt{1+n}}$ (ubi p notat $-\frac{1}{2}$) ꝏ $1-\frac{1}{2}n+\frac{1.3}{2.4}nn-\frac{1.3.5}{2.4.6}n^3+\frac{1.3.5.7}{2.4.6.8}n^4-$ &c. & pariter in cæteris.

Nota, quòd exaltatio binomii ad potestatem indefinitam & interpolationis negotium revera in idem recidunt, unoque & eodem nituntur fundamento, quod consistit in proprietate quadam numerorum figuratorum supra jam prælibata Propos. XIX. sed cujus demonstrationem, ne hic nimii simus, in aliam occasionem reservamus.

L V. *Duarum quantitatum indeterminatarum relationem unius ad alteram per seriem exprimere, ope assumtæ seriei fictæ instar quæsitæ.*

Ponatur alterutra indeterminatarum x & y, quarum relatio ad se invicem quæritur, putà y, æquari seriei $a+bx+cxx+ex^3$ $+fx^4$, &c. aut $a+bxx+cx^4+ex^6$, &c. aut $a+bx^4+cx^8$ $+ex^{12}$, &c. aut simili, prout opus videbitur; atque tum in quantitate vel æquatione proposita loco y substituatur hæc series, nec non loco dy & ddy, &c. seriei differentiale aut differentio-differentiale, &c. quo facto ex comparatione homologorum terminorum determinari poterunt assumti coëfficientes a, b, c, &c. Sequuntur Exempla:

L V I. *Invenire relationem coordinatarum Curvæ Elasticæ per seriem.* Fig. 7.

Flectatur Elater in curvaturam AQR à potentia applicata in A, & trahente juxta directionem AZ; sitque AB vel RZ ꝏ a, AE vel PQ ꝏ x, AP vel EQ ꝏ y, & AQ ꝏ z; ostensum est in Act. Lips. 1694. p. 272. & 1695. p. 538. naturam hujus curvæ exprimi æquatione dy ꝏ $\frac{xxdx}{\sqrt{a^4-x^4}}$, è qua qui methodo Diophanti, qua in præced. part. usi fuimus, irrationalitatem tollere vellet, ætatem consumeret; cum deprehensum sit à Geometris, summam vel differentiam duorum biquadratorum, qualis est a^4-x^4, nunquam posse constituere qua-

Pp dratum:

dratum: Quare nobis confugiendum est vel ad Interpolationes, vel ad indefinitam Potentiam binomii, hoc pacto:

1. *Mod.* Interpretemur x^4 tam per l, quàm per n, & a^4 per m; erit $\frac{x4}{a4-x4} \infty \frac{l}{m-n}$; unde per LIII habetur $\sqrt{\frac{l}{m-n}}$, id est, $\sqrt{\frac{x4}{a4-x4}}$ aut $\frac{xx}{\sqrt{a4-x4}} \infty \frac{xx}{aa} + \frac{1x^6}{2a^6} + \frac{1.3x^{10}}{2.4a^{10}} + \frac{1.3.5x^{14}}{2.4.6a^{14}} + \&c.$ & (facta multiplicatione per dx) $\frac{xxdx}{\sqrt{a4-x4}}$ seu $dy \infty \frac{xxdx}{aa} + \frac{1x^6dx}{2a^6} + \frac{1.3x^{10}dx}{2.4a^{10}} + \frac{1.3.5x^{14}dx}{2.4.6a^{14}}$ &c. & denique summando, AP seu $y \infty \frac{x3}{3aa} + \frac{1x7}{2.7a^6} + \frac{1.3x^{11}}{2.4.11a^{10}} + \frac{1.3.5x^{15}}{2.4.6.15a^{14}} + \&c.$

2. *Mod.* Explicemus nunc a per 1, & $-x^4$ per n; erit $a^4 - x^4 \infty 1+n$, & $\frac{1}{\sqrt{a4-x4}} \infty \frac{1}{\sqrt{1+n}}$: unde per LIV. fit $\frac{1}{\sqrt{1+n}}$ seu $\frac{1}{\sqrt{a4-x4}} \infty 1 + \frac{1}{2}x^4 + \frac{1.3}{2.4}x^8 + \frac{1.3.5}{2.4.6}x^{12} + \&c.$ & (multiplicand. per $xxdx$) $\frac{xxdx}{\sqrt{a4-x4}} \infty xxdx + \frac{1}{2}x^6dx + \frac{1.3}{2.4}x^{10}dx + \frac{1.3.5}{2.4.6}x^{14}dx + \&c.$ & integrando, $\frac{1}{3}x^3 + \frac{1}{2.7}x^7 + \frac{1.3}{2.4.11}x^{11} + \frac{1.3.5}{2.4.6.15}x^{15} + \&c.$ seu denique supplendo unitatem, $\frac{x3}{3aa} + \frac{1x7}{2.7a6} + \frac{1.3x^{11}}{2.4.11a^{10}} + \frac{1.3.5x^{15}}{2.4.6.15a^{14}} + \&c.$ ut anteà.

Coroll. Sumta $x \infty a \infty 1$, fit tota $AZ \infty \frac{1}{3} + \frac{1}{2.7} + \frac{1.3}{2.4.11} + \frac{1.3.5}{2.4.6.15} + \&c.$ Conf. Act. Lips. 1694. p. 274. & 369.

LVII. *Rectificare eandem Curvam seriem.* Fig. 7.

Quia æquatio curvæ, ut dictum, est $dy \infty \frac{xxdx}{\sqrt{a4-x4}}$, fiet quadrando $dy^2 \infty \frac{x4dx^2}{a4-x4}$ & $dz^2 \infty dy^2 + dx^2 \infty \frac{x4dx^2}{a4-x4} + dx^2 \infty \frac{a4dx^2}{a4-x4}$, adeoque $dz \infty \frac{aadx}{\sqrt{a4-x4}}$. Exponamus a^4 nunc per l, nunc per m, & x^4 per n, erit $\frac{aa}{\sqrt{a4-x4}}$ seu $\sqrt{\frac{a4}{a4-x4}} \infty \sqrt{\frac{l}{m-n}}$; unde per

per LIII. fit $\sqrt{\frac{l}{m-n}}$ sive $\sqrt{\frac{a4}{a4-x4}} \infty 1 + \frac{1x4}{2a4} + \frac{1.3x^8}{2.4a^8} + \frac{1.3.5x^{12}}{2.4.6a^{12}} + \&c.$ & (multiplic. per dx) $\frac{aadx}{\sqrt{a4-x4}}$ seu $dz \infty dx + \frac{1x4dx}{2a4} + \frac{1.3x^8dx}{2.4a^8} + \frac{1.3.5x^{12}dx}{2.4.6a^{12}} + \&c.$ tandemque summando, z sive $AQ \infty x + \frac{1x5}{2.5a4} + \frac{1.3x9}{2.4.9a8} + \frac{1.3.5x^{13}}{2.4.6.13a^{12}} + \&c.$ Idem etiam per LIV. simili modo ostendetur.

Coroll. Facta $x \infty a \infty 1$, habetur tota $AQR \infty 1 + \frac{1}{2.5} + \frac{1.3}{2.4.9} + \frac{1.3.5}{2.4.6.13} + \&c.$ vid. Act. Lips. 1694. p. 274.

LVIII. *Definire limites præcedentium serierum.* Fig. 7.

Quoniam series his methodis repertæ nimis lentè convergunt, non abs re erit, si modum ostendam, quo levi labore summis earum, quantum ad usum sufficit, approximare & limites constituere possimus. In exemplum propositæ sint proximæ duæ series, quibus exprimitur applicata Elasticæ BR vel AZ, & longitudo ipsius curvæ AR, nempe: $\frac{1}{3} + \frac{1}{2.7} + \frac{1.3}{2.4.11} + \frac{1.3.5}{2.4.6.15} + \&c.$ &, $1 + \frac{1}{2.5} + \frac{1.3}{2.4.9} + \frac{1.3.5}{2.4.6.13} + \&c.$ Sumo quantitatem, cujus integrale haberi possit, datis $\frac{xxdx}{\sqrt{a4-x4}}$ & $\frac{aadx}{\sqrt{a4-x4}}$, è quibus series propositæ fluxerunt, affinem, putà $\frac{x3dx}{\sqrt{a4-x4}}$, cujus integrale est $\frac{aa-\sqrt{a4-x4}}{2}$, eamque pari methodo in seriem resolvo, & seriei terminis summatis pro x & a unitatem pono; quo pacto series emerget: $\frac{1}{4} + \frac{1}{2.8} + \frac{1.3}{2.4.12} + \frac{1.3.5}{2.4.6.16} + \&c.$ æqualis proinde $\frac{aa-\sqrt{a4-x4}}{2} \infty \frac{1}{2}$ seu 0. 5000000. Colligo jam singularum serierum terminos aliquot ab initio in unam summam (quod expeditè fit per Logarithmos) ex. gr. decem primos terminos, qui collecti efficiunt in prima serie 0. 5102560: in secunda serie 1. 2207187: in tertia 0. 4119014. Hujus igitur reliqui post decimum

decimum termini (ad complendum $\frac{1}{2}$ seu 0. 5000000) constituent 0. 0880986, qui numerus additus summæ 10 primorum terminorum in pr. & sec. serie exhibet 0. 5983546 & 1. 3088173, summis totarum serierum justo minores, ob singulos tertiæ seriei terminos minores homologis terminis reliquarum.

Deinde, quia undecimi termini in tribus istis seriebus sunt $\frac{1.3.5.7.9.11.13.15.17.19}{2.4.6.8.10.12.14.16.18.20.43}$, $\frac{1.3.5\ldots\ldots19}{2.4.6\ldots20.41}$, $\frac{1.3.5\ldots\ldots19}{2.4.6\ldots20.44}$, liquet terminum hunc in serie tertia ad eundem in serie prima reciprocè esse ut 43 ad 44, & ad eundem in secunda ut 41 ad 44; terminorum verò sequentium singulos in tertia serie ad ejusdem ordinis terminos in reliquis seriebus habere rationem majorem quàm 43 ad 44, & quàm 41 ad 44: unde & summa omnium sequentium decimum in tertia serie ad summam omnium post decimum in reliquis seriebus majorem rationem habebit. Idcircò si fiat, ut 43 ad 44, nec non ut 41 ad 44, ita summa terminorum post decimum in tertia serie, nimirum 0. 0880986, ad 0. 0901474 & ad 0. 0945448; erunt hi numeri majores summis terminorum decimum sequentium in prima & secunda serie: quapropter si addantur summis 10 priorum, quæ sunt 0. 5102560 & 1. 2207187, erunt quoque numeri provenientes 0. 6004034 & 1. 3152635 majores summis totarum serierum.

Reperti ergo sunt limites, quibus summæ primæ & secundæ seriei definiuntur: limites illius sunt 0. 5983546 & 0. 6004034; hujus 1. 3088173 & 1. 3152635: unde applicata BR vel AZ major est quàm 0. 598, & minor quàm 0. 601; ipsa verò curva AR > 1. 308, & < 1. 316, sic ut tres istæ lineæ RZ, AZ & AQR proximè se habeant ut 10, 6, 13. Conf. Act. Lips. 1694. p. 274.

Schol. Quoniam ex natura descensus gravium demonstratur, quòd tempus descensus penduli alicujus per quadrantem circuli ad tempus descensus perpendicularis per ejus radium eam rationem habet, quam habet Curva Elastica AR ad ejus axem RZ, h. e. majorem, ut ostendimus, quàm 1308 ad 1000, & minorem quàm 1316 ad 1000: tempus autem descensus perpendicularis per circuli radium ad tempus per semiradium se habet, ut $\sqrt{2}$ ad 1: & tempus per semiradium

ad tempus per arcum minimum (consentiente Hugenio in Horol. Oscillat. p. 155.) ut diameter circuli ad ejus semiperipheriam, h. e. ut 226 ad 355: inferri potest ex æquo, quòd tempus descensus penduli per quadrantem integrum ad tempus descensus ejus per arcum minimum se habet in ratione majore quàm 3400 ad 2888, & in minore quàm 3400 ad 2869; unde rationem 3400 ad 2900, sive 34 ad 29, quam præfatus Auctor ibid. pag. 9. temporibus horum descensuum assignat, extra hos limites cadere liquet.

LIX. *Dati Logarithmi Numerum invenire per seriem.* Fig. 1.

Intelligatur Curva Logarithmica PCQ, cujus axis AD, subtangens constans $\infty\, t$, applicata $BC \infty 1$, Logarithmus datus BI ($B\iota$) $\infty\, x$, ejusque Numerus IO (ιo) $\infty\, y$; erit ex generali curvarum natura $\pm\, dy . dx :: y . t$, adeoque $y \infty \pm \frac{tdy}{dx}$. Fiat juxta præscriptum Prop. LV. $y \infty 1 + bx + cxx + ex^3 + fx^4$ &c. & differentiando, $\frac{dy}{dx} \infty b + 2cx + 3exx + 4fx^3$ &c. eritque $1 + bx + cxx + ex^3 + fx^4$, &c. $(\infty\, y \infty \pm \frac{tdy}{dx}) \infty \pm bt \pm 2ctx \pm 3etxx \pm 4ftx^3 \pm 5gtx^4$, &c. & facta comparatione homologorum terminorum elicietur, $b \infty \pm \frac{1}{t}$, $c\ (\infty \pm \frac{b}{2t}) \infty \frac{1}{1.2.tt}$, $e\ (\infty \pm \frac{c}{3t}) \infty \pm \frac{1}{1.2.3t^3}$, $f\ (\infty \pm \frac{e}{4t}) \infty \frac{1}{1.2.3.4t^4}$, &c. unde valoribus istis coëfficientium b, c, e, &c. substitutis resultat $y \infty 1 \pm \frac{x}{t} + \frac{xx}{1.2tt} \pm \frac{x^3}{1.2.3t^3} + \frac{x^4}{1.2.3.4t^4} \pm$ &c. Conf. Act. Lips. 1693. p. 179.

Aliter idem absque differentialium adminiculo: Concipiatur Log-us BI ($B\iota$) divisus in partes quotlibet æquales BE, EF, FG, &c. ($B\epsilon$, $\epsilon\phi$, $\phi\gamma$, &c.) quarum numerus sit n, & singulæ dicantur d, sic ut nd fit ∞ BI ($B\iota$) $\infty\, x$. Tum applicatis curvæ rectis totidem EK, FL, GM, &c. ($\epsilon\kappa$, $\phi\lambda$, $\gamma\mu$, &c.) jungantur extremitates C & K (κ) duarum BC, EK ($\epsilon\kappa$) per rectam CK ($C\kappa$), sitque axis portio inter productam CK ($C\kappa$) & applicatam BC intercepta $\infty\, t$; quo pacto propter triangula similia fiet $t . 1\ (BC) :: t \pm d . 1 \pm \frac{d}{t} \infty EK$ ($\epsilon\kappa$). Et

 quoniam

quoniam ob æquales BE, EF, FG, &c. ($B\epsilon, \epsilon\phi, \phi\gamma$, &c.) ipsæ BC, EK, FL, &c. ($BC, \epsilon\kappa, \phi\lambda$, &c.) in continua sunt proportione, earumque prima $BC \infty 1$, idcircò designabit FL ($\phi\lambda$) secundam potestatem, GM ($\gamma\mu$) tertiam, RN ($\varrho\nu$) quartam, &c. tandemque ultima IO (ιo) seu y ipsam n potestatem applicatæ EK ($\epsilon\kappa$) seu $1\ 8\ \frac{d}{t}$ (pro numero vid. particularum, in quas divisa est BI ($B\iota$); quæ quidem potestas per LIV. reperitur $\infty\ 1\ 8\ \frac{nd}{t} + \frac{n.\overline{n-1}.dd}{1.2tt}\ 8\ \frac{n.\overline{n-1}.\overline{n-2}d^3}{1.2.3t^3} + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.d^4}{1.2.3.4t^4}\ 8$ &c. Quòd si jam numerus particularum n ponatur infinitus, producta CK ($C\kappa$) abibit in tangentem, & ipsa t in subtangentem Logarithmicæ, atque prætereà numeri 1, 2, 3, &c. evanescent præ n, sic ut $n-1$, $n-2$, $n-3$, tantundem valeant ac n: quare tum fiet $y \infty 1\ 8\ \frac{nd}{t} + \frac{nndd}{1.2tt}\ 8\ \frac{n^3d^3}{1.2.3t^3} + \frac{n^4d^4}{1.2.3.4t^4}\ 8$ &c. ∞ (propter $nd \infty x$) $1\ 8\ \frac{x}{t} + \frac{xx}{1.2tt}\ 8\ \frac{x^3}{1.2.3t^3} + \frac{x^4}{1.2.3.4t^4}\ 8$ &c. ut suprà.

Nota, quòd existente $x > t$, termini quidem seriei aliquousque crescunt, tandem tamen decrescere pedetentim occipiunt, ultimoque vergunt in nihilum. Nam sumtis ab initio m terminis, erit ex lege progressionis sumtorum ultimus $\frac{x^{m-1}}{1.2.3\ldots\overline{m-1}.t^{m-1}}$, & sequens ultimum $\frac{x^m}{1.2.3\ldots m t^m}$; adeoque ratio illius ad hunc, ut mt ad x: unde cum ratio t ad x determinata sit, numerus verò terminorum m usque & usque major possit accipi, ratio quoque mt ad x tandem quavis data major fiet. Existente autem $x \infty$ vel $< t$, series ista, & aliæ hujus generis, statim ab initio celerrimè convergunt, eoque celeriùs quò minor x: unde discimus quod multo commodius & minori cum labore Logarithmorum Canon adornari possit, si per hanc Propos. ex Log-is datis Numeri, quàm si vicissim per XLVII. ex Numeris datis Log-i quærantur. Quanquam & illic compendium sese nobis offerat non contemnendum, quod quia in dicta Propos. intactum præteriit, breviter hic indicandum restat: Quoniam positis in Logarithmica (Fig. 6.) $AB \infty a$, subtang. $AK \infty t$, $BI \infty u$, & $B\iota \infty s$, adeoque $RE \infty a - u$, & $\varrho\epsilon \infty a + s$, invenitur per XLVII, AR (Log-us RE) ∞t in $\overline{\frac{u}{a} + \frac{uu}{2aa} + \frac{u^3}{3a^3} + \frac{u^4}{4a^4} + \&c.}$ & $A\varrho$ (Log-us $\varrho\epsilon$) ∞t in $\overline{\frac{s}{a} - \frac{ss}{2aa} + \frac{s^3}{3a^3} - \frac{s^4}{4a^4} + \&c.}$ sequitur ex natura Log-micæ, has duas series inter se æquari, si tres applicatæ $RE, AB, \varrho\epsilon$, seu, $a-u$, a & $a+s$ continuè proportionentur, h. e. si statuatur $u \infty \frac{as}{a+s}$; sed quia per hanc hypothesin perpetuo fit $u < s$, & nominatim hac sumta $\infty a, \frac{1}{2}a, \frac{1}{3}a$, &c. illa fit $\infty \frac{1}{2}a, \frac{1}{3}a, \frac{1}{4}a$, &c. multò semper celerius prior series converget posteriore: unde plurimum laboris in practica effectione log-orum rescindi poterit, si loco hujus illa surrogetur, ex. gr. si (facta $s \infty a$) loco seriei $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$ &c. hoc est, loco $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} +$ &c. substituatur $\frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} +$ &c. quippe per cujus primos 18. terminos tantundem approximatur, quantum per mille terminos alterius; quod ipsum etiam ad Coroll. 3. XLVII. in subtangente Log-micæ definienda observabitur. Sed rei utilissimæ uberiorem explicationem angustia paginæ non permittit.

Schol. Si summa quædam pecuniæ fœnori elocata sit, ea lege, ut singulis momentis pars proportionalis usuræ annuæ in sortem computetur; exponatur autem ipsa sors per BC seu 1, tempus annuum per BI seu x divisum in punctis E, F, G, &c. in momenta innumera æqualia, atque usura annua per $\frac{x}{t}$; inventa series $1 + \frac{x}{t} + \frac{xx}{1.2tt} + \frac{x^3}{1.2.3t^3} +$ &c. hoc est, (explicata sorte 1 per a, & usura $\frac{x}{t}$ per b) $a + b + \frac{bb}{2a} + \frac{b^3}{2.3aa} + \frac{b^4}{2.3.4a^3} +$ &c. indicabit valorem ejus, quod finito anno debebitur. Cum enim, ut tempus annuum BI ad primum ejus momentum BE, seu ut x ad d, ita se habeat usura annua $\frac{x}{t}$ ad partem proportionalem usuræ, erit hæc $\frac{d}{t}$, signifi-

ſignificabitque $1 + \frac{d}{t}$ ſeu applicata EK ſortem dicta parte proportionali uſuræ auctam: unde ſors aucta EK ſecundo momento pariet FL, & hæc pariter tertio momento pariet GM, & ſic porrò, propter BC, EK, FL, GM, &c. $\div$. Quare poſtrema applicata IO, quam ſeries inventa exprimit, denotabit valorem ejus, quod creditori elapſo toto anno debetur. Conf. Act. Lipſ. 1690. p. 222.

LX. *Invenire aream ſpatii comprehenſi à Curva genitrice Elaſtica, ſeu qua evolutione ſui Elaſticam deſcribit.* Fig. 7.

Deſcribatur Elaſtica AQR ex evolutione Curvæ MNT, & ſit filum evolvens QN (DG), quod productum ſecet axem in V; ponaturque, ut ſupra, $RZ \infty a$, $PQ \infty x$, $AP \infty y$. Quoniam ex Act. Lipſ. 1694. p. 273. manifeſtum eſt, quòd $QN \infty \frac{1}{2} QV$, erit & $NH \infty \frac{1}{2} PQ \infty \frac{1}{2} x$, & $NS \infty \frac{1}{2} FQ \infty \frac{1}{2} dx$; ac proinde ob ang. rect. DQN, $DF . FQ$ (:: $dy . dx$ :: [ex natura Elaſticæ] $xx . \sqrt{a^4 - x^4}$) :: $\frac{1}{2} dx (NS) . \frac{dx\sqrt{a^4-x^4}}{2xx} \infty SG$ vel HI. Quare HI in NH ſeu $\square NI \infty \frac{xdx\sqrt{a^4-x^4}}{4xx} \infty \frac{\overline{a^4x - x^5}, dx}{4xx\sqrt{a^4-x^4}} \infty \frac{a^4xdx}{4xx\sqrt{a^4-x^4}} - \frac{x^3dx}{4\sqrt{a^4-x^4}} \infty$ Elemento ſpatii $MNHZ$, de cujus ſummatione jam agitur. Poſterioris membri $\frac{x^3dx}{4\sqrt{a^4-x^4}}$ integrale pertinens ad partem curvæ RQ vel MN eſt $\frac{1}{8}\sqrt{a^4 - x^4}$. Prius autem $\frac{a^4xdx}{4xx\sqrt{a^4-x^4}}$ cum abſolutè ſummari nequeat, ſublata irrationalitate in ſeriem convertetur, ut ſequitur.

Ponatur $\sqrt{a^4 - x^4} \infty \frac{txx}{a} - aa$, fiet $xx \infty \frac{2a^3t}{aa+tt}$, & differentiando $-xdx \infty \frac{\overline{a^3tt - a^5}, dt}{Q: aa+tt}$; nec non $\frac{txx}{a} - aa$ $(\sqrt{a^4-x^4}) \infty \frac{aatt - a^4}{aa+tt}$, & denique $\frac{-a^4xdx}{4xx\sqrt{a^4-x^4}} \infty \frac{aadt}{8t}$. Jam quia exiſtente maxima $x \infty a$, ipſa quoque $t \infty a$, & illa decreſcente creſcit hæc, ſtatuatur

ſtatuatur $t \infty a + s$, ut ſit $\frac{aadt}{8t} \infty \frac{aads}{8a+8s} \infty \frac{aa}{8}$ in $\frac{ds}{a+s} \infty \frac{aa}{8}$ in $\overline{\frac{ds}{a} - \frac{sds}{aa} + \frac{ssds}{a^3} - \frac{s^3ds}{a^4} + \text{\&c.}}$ per XXXVII: unde facta ſummatione habetur $S\frac{aadt}{8t}$ (∞ $S\frac{a^4xdx}{4xx\sqrt{a^4-x^4}}$, diſſimulato nempe ſigno $-$, quod hic nota tantum eſt reſpectivi decrementi ipſarum x) $\infty \frac{aa}{8}$ in $\overline{\frac{s}{a} - \frac{ss}{2aa} + \frac{s^3}{3a^3} - \frac{s^4}{4a^4} + \text{\&c.}}$ demtoque $S\frac{x^3dx}{4\sqrt{a^4-x^4}} \infty \frac{1}{8}\sqrt{a^4-x^4}$, reſultat $S\frac{a^4xdx}{4xx\sqrt{a^4-x^4}} - S\frac{x^3dx}{4\sqrt{a^4-x^4}} \infty \frac{aa}{8}$ in $\overline{\frac{s}{a} - \frac{ss}{2aa} + \frac{s^3}{3a^3} - \frac{s^4}{4a^4} + \text{\&c.}} - \frac{1}{8}\sqrt{a^4-x^4}$ ſpatio nempe quæſito $MNHZ$. Et quia, ſumta $u \infty \frac{as}{a+s}$, ſeries $\frac{s}{a} - \frac{ss}{2aa} + \frac{s^3}{3a^3} -$ &c. æquatur ſeriei $\frac{u}{a} + \frac{uu}{2aa} + \frac{u^3}{3a^3} + \frac{u^4}{4a^4} +$ &c. per Annot. præc. Propoſ. idcirco dictum ſpatium $MNHZ$ quoque ſic exprimetur, $\frac{aa}{8}$ in $\overline{\frac{u}{a} + \frac{uu}{2aa} + \frac{u^3}{3a^3} + \frac{u^4}{4a^4} + \text{\&c.}} - \frac{1}{8}\sqrt{a^4-x^4}$.

Nota, ſi ſtatuantur $aa \infty 8$, & $s \infty a$, adeoque t $(a+s) \infty 2a$, & x $\left(\sqrt{\frac{2a^3t}{aa+tt}}\right) \infty 2a\sqrt{\frac{1}{5}}$, & u $\left(\frac{as}{a+s}\right) \infty \frac{1}{2}a$: hoc eſt, ſi conſtructo ſuper MZ, ſemiſſe ipſius RZ, ſemicirculo inſcribatur Triangulum Iſoſceles MCZ, cujus crus MC unitatem deſignet, atque Curvæ MNT applicetur NH $(\frac{1}{2}x) \infty \sqrt{\frac{8}{5}}$, prædictum ſpatium $MNHZ$ fiet $\infty 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$ &c. $- \frac{3}{5}$. vel etiam $\infty \frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} +$ &c. $- \frac{3}{5}$. Conf. Act. Lipſ. 1694. p. 273.

Coroll. 1. Quoniam ex iis, quæ loc. modo cit. Actorum docuimus, colligi poteſt, quod $QV \infty \frac{aa}{x}$, & $QN \infty \frac{1}{2} QV \infty \frac{aa}{2x}$, & DQ ſeu

DQ seu $dz \propto \frac{aadx}{\sqrt{a^4 - x^4}}$; sequitur, triangulum QGD (QD in $\frac{1}{2} QN$) $\propto \frac{a^4 dx}{4x\sqrt{a^4 - x^4}}$, & per consequens omnia triangula QGD seu spatium $RMNQR \propto S\frac{a^4 dx}{4x\sqrt{a^4 - x^4}} \propto$ (ut ostensum) spatio $MNHZ + S\frac{x^3 dx}{4\sqrt{a^4 - x^4}}$: unde cum $S\frac{x^3 dx}{4\sqrt{a^4 - x^4}}$ seu $\frac{1}{8}\sqrt{a^4 - x^4}$ exprimat quadrantem spatii Elastici $PQRZ$ (ut per se liquet), concludimus, spatium $RMNQR$ excedere aream $MNHZ$ quarta parte ipsius $PQRZ$.

Coroll. 2. Quia differentiale $\frac{aadt}{8t}$, ad quod reduximus elementum spatii $MNHZ$ vel $RMNQR$, elementum quoque denotat spatii hyperbolici inter asymptotas, cujus abscissa à centro est $\propto t$, ipsa vero t in assumta hypothesi $\sqrt{a^4 - x^4} \propto \frac{txx}{a} - aa$ propter x decrescentem ad nihilum excrescat in infinitum, & spatium hyperbolicum in infinitum protensum sit infinitum; idcirco & spatium totum interminatum genitricis Elasticæ $MNTXZ$ seu $NTXH$ infinitum erit. Vid. Act. Lips. loc. cit.

UT non-finitam Seriem finita coërcet,
Summula, & in nullo limite limes adest:
Sic modico immensi vestigia Numinis hærent
Corpore, & angusto limite limes abest.
Cernere in immenso parvum, dic, quanta voluptas!
In parvo immensum cernere, quanta, Deum!

LETTRE

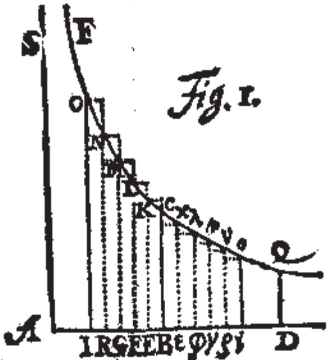
Fig. 1.
S
F
Q
A
D

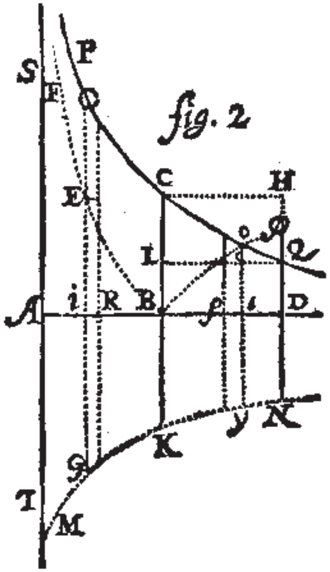
fig. 2
S
P
F
E
C
H
L
Q
A
i
R
B
D
K
N
G
T
M

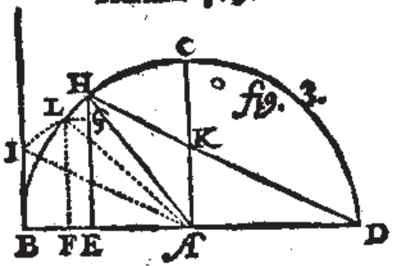
fig. 3.
C
H
L
K
I
B
F
E
A
D

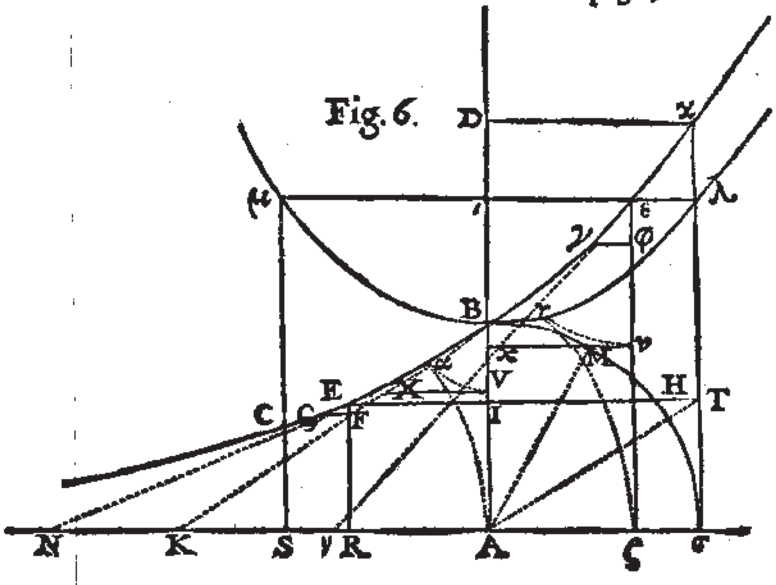
Fig. 6.
D
B
E
C
F
H
T
I
V
N
K
S
R
A

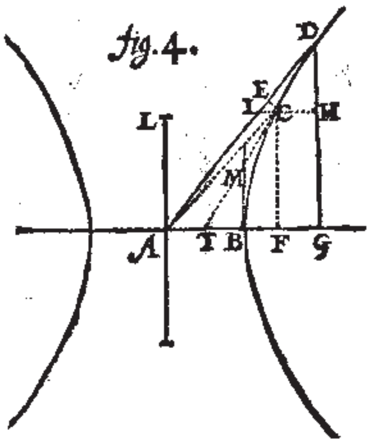
fig. 4.
D
E
I
C
M
L
A
T
B
F
G

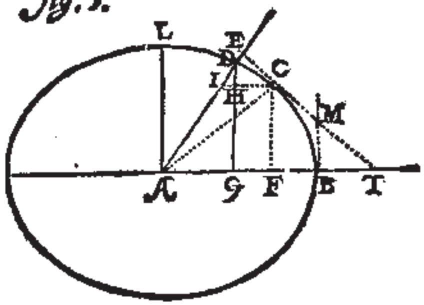
fig. 5.
L
E
D
C
I
H
M
A
G
F
B
T

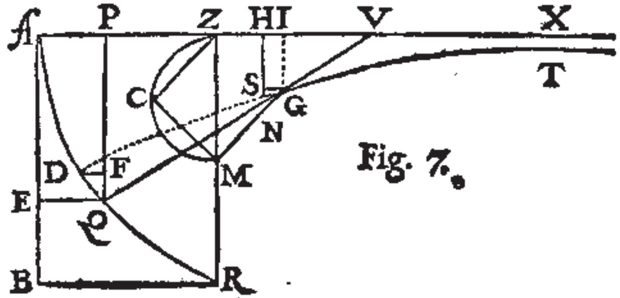
Fig. 7.
A
P
Z
H
I
V
X
T
S
C
G
N
D
F
M
E
Q
B
R